Validating Valuation: How Statistical Learning Can Cabin Expert Discretion in Valuation Disputes

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This article challenges conventional methods used in financial valuation across transactional and litigation domains. We show that conventional valuation methods allow for considerable discretion, making it possible for each side's experts to submit dramatically varying valuations simply by choosing among facially reasonable values of parameters that must be selected to carry out conventional valuations. We use large-scale empirical simulations powered by real-world data to demonstrate the scope of such discretion. We next consider several alternatives based on data-driven machine learning approaches, and show that they offer both approximately unbiased estimates of valuation and substantially reduced variability in valuation results. Consequently, they reduce the scope of expert or party discretion in valuation. We end by applying this approach to a well-known Delaware valuation dispute.

This paper thus both diagnoses and offers a remedy for the discretion and variability in valuation disputes. Although we focus our principal analysis on the comparable companies approach, many of the insights we develop here lend themselves to other valuation methodologies (such as comparable transactions and discounted cash flow analysis). If adopted, our methods would lead to better performing and more empirically grounded outcomes in legal disputes involving valuation, thus enhancing the fairness and efficiency of the judicial processes in valuation litigation.

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I. INTRODUCTION

For the past four decades, financial valuation has played an increasingly pivotal role in high-stakes business law practice. While the law's embrace of such methodologies was initially confined to isolated topics in corporate and securities law, the practice quickly expanded. Legal and financial practitioners work hand-in-hand in designing and executing transactions. Perhaps, consequently, commercial litigation has come to be influenced by valuation disputes that hinge on financial economics—spanning a range of practice areas from bankruptcy, to tax, to family law, to fiduciary duties, to garden-variety questions in tort, property, and contract law. Every top U.S. law school now offers at least one course dedicated to teaching financial valuation techniques to law students.

However, the wholesale adoption of financial valuation in commercial litigation has had unfortunate collateral consequences. The first stems from the fact that financial economics tends to be a mathematical enterprise. Most judges and legal practitioners are not formally trained asset-pricing specialists, even in business-intensive jurisdictions; yet they are now typically required to process, understand, admit, exclude, and sometimes weigh technical financial evidence, and often to instruct lay juries possessing even less expertise. Judges and practitioners who lack technical training themselves are left to pick up key tenets of valuation practice on the fly.

Second, despite its seemingly precise technical façade, financial valuation is awash with free parameters that require judgment—thereby affording expert analysts substantial practical discretion. Meanwhile, the professional literature meant to guide and substantiate financial experts' choices includes a variety of would-be "authoritative manuals" that contain distinct, arcane, and sometimes inconsistent formulations for best practices.[1] Such factors amplify the need for experts who can exercise such discretion in an informed manner.

Third, as is customary in the American adversarial system, courtroom financial experts are typically compensated by each side's litigants. Although many of these experts are respected members of academic or professional finance circles, their consulting work involves repeat play, and expert reports and deposition testimony are often largely shielded from public scrutiny.

1. *E.g.*, IVO WELCH, CORPORATE FINANCE (5th ed. 2022).

Although evidence law prohibits cherry-picking by an expert,[2] there are limits to the observability of what experts do with the data they use, and calculations left on the cutting room floor may be impossible for others to reconstruct. Expert reports, moreover, are often not filed on a case's docket, or are filed under seal, which keeps them out of public view. Experts might thus be tempted to embrace techniques in litigation that they would avoid, or even criticize, in academic settings.[3]

The final problem arises because judges who adjudicate claims of dueling financial experts generally must provide reasons for their judgments in their written opinions. The natural course for a non-expert judge is to adopt a particular expert's rationale for each key aspect of a decision. The sum of such parts may be an incoherent whole. And even if not, once a judge's reasoning, whether right or wrong, is memorialized in a written decision, it can become reified through the common law process, thereafter, serving as persuasive authority, or even binding precedent, that validates experts' discretionary choices. Such "rulified" precedent, in turn, not only might affect future cases, but also might distort transactional procedures in those cases where parties anticipate the prospect of later adjudication.

In recent years, courts have undertaken piecemeal reforms to address these issues. For example, some have embraced the practice of routinely unsealing expert reports for public scrutiny, at least after trial, under the theory that the prospect of professional embarrassment can provide needed discipline.[4] Others have suggested structural reforms, such as having courts retain an independent expert to advise on valuation issues,[5] or experimenting with expert "hot-tubbing",[6] or committing to final-offer arbitration mechanisms to incentivize experts towards moderation.[7] Still others have attempted to sidestep altogether the messy enterprise of valuation sausage making, by, for example, simply adverting to negotiated prices in merger cases, or to pre-existing securities market prices.[8]

2. *See, e.g.*, FED. R. EVID. 702(b) (requiring that expert testimony be "based on sufficient facts or data," which can be understood to prevent cherry-picking); Jonah B. Gelbach, *Estimation Evidence*, 168 U. PA. L. REV. 549, 587 (2020).

3. *See, e.g.*, *In re* SWS Grp., Inc., No. 10554, 2017 WL 2334852, at *11–12 (Del. Ch. May 30, 2017); *see also* Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd., 177 A.3d 1, 36 (Del. Dec. 14, 2017) ("As is common in appraisal proceedings, each party—petitioners and the Company—enlisted highly paid, well-credentialed experts to produce DCF valuations. But their valuations landed galaxies apart—diverging by approximately $28 billion, or 126%.").

4. *See* Eric L. Talley, *Finance in the Courtroom: Appraising Its Growing Pains*, 35 DEL. LAW. 16, 17 (2017) (discussing the prohibition on "judicial use of incentive devices").

5. Andrew M. Smith, *Using Impartial Experts in Valuations: A Forum-Specific Approach*, 35 WM. & MARY L. REV. 1241, 1241–43 (1994).

6. Hot-tubbing—also sometimes referred to as "expert day"—involves having the experts interact with one another directly, such as by simultaneously appearing for questions by the judge or through mandated authoring a joint report. *See* Dan Papscun, *Courtroom 'Hot Tub' Puts Google Trial Experts to Stress Test*, BLOOMBERG L. (Oct. 6, 2023), https://news.bloomberglaw.com/antitrust/courtroom-hot-tub-puts-google-trial-experts-to-stress-test (on file with the *Journal of Corporation Law*).

7. *See* Keith Sharfman, *Valuation Averaging: A New Procedure for Resolving Valuation Disputes*, 88 MINN. L. REV. 357, 365–66 (2003) (detailing the "final offer" approach).

8. *See* Verition Partners Master Fund Ltd. v. Aruba Networks, Inc., No. 11448, 2018 WL 2315943, at *2 (Del. Ch. May 21, 2018) ("Under a traditional formulation of the efficient capital markets hypothesis, the unaffected market price provides a direct indication of the value of the subject company based on its operative reality independent of the merger, at least for a company that is widely traded and lacks a controlling stockholder. I therefore concluded on the facts presented that the most persuasive evidence of Aruba's fair value was its unaffected trading price . . ."), *rev'd*, 210 A.3d 128 (Del. 2019); *Magnetar Glob. Event Driven Master Fund Ltd*, 177

This article pursues a different path, one rooted in the basics of financial valuation. We propose that courts adjudicating valuation disputes should require experts to demonstrate that their discretionary choices actually lead to better predictions of company value. We point out that contemporary methods, including the so-called comparable companies (CC) approach, can be viewed as an instance of what is known in the machine-learning field as nearest-neighbor matching. As they are currently applied, these approaches have numerous "expert degrees of freedom." For example: How many comparable companies will be included? How is comparability to be determined? How are comparable companies' values used to place a value on the company at issue in the litigation? We show further that these expert degrees of freedom can be expected to lead to sizable gaps between opposing experts' valuations in litigation. Importantly, all of this is true *even if experts use widely accepted methods*; the discretion emanates from the need to exercise judgment about *how* to use such methods.

Our central argument stems from a fundamental observation: at their core, valuation methodologies are nothing more than exercises in prediction about the fair market value of an asset. This statement remains true regardless of whether the methodology is used to estimate the fair value of a company, security, land, equipment, contract right, business opportunity, bankruptcy claim, intellectual property, and more. The key insight here is that valuation methodologies are worth deploying only if there is reason to think they will render a credible prediction of the target asset's value given available data.

As we demonstrate below, each of the three leading market valuation methodologies currently in use in legal settings—CC analysis, comparable transactions (CT) analysis, and discounted cash flow (DCF) analysis—can be viewed as wholly or partly involving the logic of what is known in the machine learning literature as k-nearest neighbor (k-NN) prediction. Given that each approach aspires to deliver a prediction of a target firm's value on a target date, these methodologies differ only in the choice and use of data used to accomplish the task of prediction. In the pages below, we put these observations to work (a) illustrating the current problem concretely, and (b) proposing a promising and practical solution. We illustrate both claims with a series of statistical simulations that use actual public company data. We focus on the comparable companies approach, but our framework generalizes to other methodologies.

First, we demonstrate not only how the CC approach entails significant discretion, but also that such discretion plausibly leads to substantial bias and variability in expert valuations. Both of these features are undesirable from an adjudication perspective, providing substantial reason to discipline further expert use of conventional valuation methods. While judicial actors can take some assertive measures within the current system to cope with this discretion, much of the problem remains even when we confine ourselves to prevailing methods.

We then turn to our constructive contribution, formulating an alternative approach that—we demonstrate—can deliver substantially more reliable valuation predictions than CC or similar methods. The alternative approaches we develop use similar raw data to

A.3d at 35 ("Taken as a whole, the market-based indicators of value—both Dell's stock price and deal price—have substantial probative value.").

those used in the CC approach.[9] But instead of being left to expert discretion, we let the data themselves drive choices about what companies to use in comparisons, how to weight data for those companies, and how to aggregate the company-specific information.[10] More specifically, we demonstrate that our approach yields predictions that are, on average, accurate when compared to the target firm's observed value as of the target date. Critically, our simulations show that our methods are all far less variable, and thus less manipulable, than the simulated outcomes under the prevailing CC method. The improvements we generate are especially strong when we use higher frequency data (daily) based on stock returns to predict various valuation measures. Given its use of daily returns, this method is available as a substitute only for CC analysis, i.e., it would not be available when comparable transactions or discounted cash flow approaches are desired for reasons other than predictive power. Still, the strong performance of our daily-returns approach brings a countervailing methodological bonus: adopting it would help increase the law's uniformity in other doctrinal domains of financial valuation, because daily stock return data are at the center of another primary financial litigation field, namely, securities fraud.

In sum, our project makes two important contributions. First, using real-world simulations with actual market valuation data, we show that current courtroom valuation practices are inherently highly variable, affording considerable expert degrees of freedom. Second, we show that market cap and/or stock returns data can usefully be used together with contemporary statistical techniques to leverage data-driven estimation, reducing expert discretion, improving the performance of valuation methods, and thus helping increase the predictability of financial litigation.

One caveat warrants consideration. If current valuation practices are cursed by the problem of too much expert discretion, one can justifiably wonder whether our proposed solution suffers from the same problem. After all, we show that many ML-driven models outperform current practice with respect to both bias and variance; how should an expert, or a judge, choose among these ML models, or even from among others? Moreover, our analysis below is largely limited to a modest set of flexible linear statistical models, and there are many other available approaches (e.g., neural networks, pretrained LLMs, and ensemble models that blend multiple approaches). If financial experts do leave behind traditional valuation methodologies in favor of data-driven approaches, what would cabin their discretion over which alternative model to select? To put a finer point on it: does our proposed cure just unleash a new strain of the original disease?

Although this criticism has some validity, we believe it is mitigated for at least two reasons. First, because our results show that ML-driven approaches systematically generate more reliable predictions than conventional practice, even an expert motivated to use discretion will have less room to find it; that is, the methods we introduce below simply result in less dramatic valuation variation. More important, though, is that once experts start using statistical models to train their predictive models,

9. The specific approaches we develop below permit the algorithm to select the best predictors from an entire industry. We also discuss variations of our model based on varying the frequency with which financial data are observed comparables (quarterly versus daily); the measure observed in the comparisons (multiples, total market capitalization and returns); and the measure for the firm whose value one is attempting to predict (multiples versus market capitalization). In each case, our approach significantly outperforms our simulation of current practice.

10. Because explainability is important in judicial applications, we use a familiar type of statistical model known as the penalized regression approach. This approach has the advantage that the most predictive set of comparable firms, measuring dates, and weighting/aggregation criteria are based on computations using an algorithm that is observable to others—including the judge—besides the expert who uses the approach.

adjudicators can become more involved and confident in making comparisons among them. A judge might require each expert to produce a replicable estimate of performance, such as the model's mean squared error, using out-of-training-sample data. The model achieving the best performance score would accordingly get more (or sole) weight in the judge's decision. Other such validation/verification methods might develop. Thus, while our approach may not completely cure the disease of expert discretion, it can be expected to bring a substantial reduction in symptoms.

Our analysis proceeds as follows. In Part II, we provide an intuitive overview of standard valuation approaches that are prevalent in legal and financial practice: comparable transactions, comparable companies, and discounted cash flow analyses. Part III argues that the comparable transactions and comparable companies methods can be viewed as a type of nearest neighbor matching algorithm.[11] In Part IV.A, we focus on the comparable companies analysis, demonstrating the extent of the expert degrees of freedom problem, using a large-scale simulation of 10,000 target firms and dates. Then in Part IV.B, we propose and evaluate a series of competing approaches using market cap and stock price data together with data-driven prediction methods. These methods soundly outperform conventional approaches. In Part V, we apply our alternative approach to the famous *DFC Global* appraisal case from Delaware—which itself produced highly divergent expert valuations—using it to show how our proposed methods work in an actual litigated case. We discuss various limitations and extensions to our analysis in Part VI.

II. CONVENTIONAL PRACTICE

As currently practiced in courtrooms and boardrooms, modern valuation practice is dominated by three alternative methodologies: comparable companies (CC), comparable transactions (CT), and discounted cash flow (DCF) analyses. In many cases, a valuation expert will attempt to value a financial asset of interest using two, or even all three, of these approaches.[12] Particularly in cases of company valuations associated with mergers or bankruptcies, experts may use other valuation methodologies as complements. Such additional approaches include an analysis of historical premiums paid, analyst forecasts, and leveraged buyout/recapitalization analysis. When such alternatives are employed, however, they are typically offered only as secondary "reference" valuations meant to complement CC, CT, and DCF.[13]

A. Comparable Transactions

We start with the CT approach, because it may be the most intuitively accessible, as it bears resemblance to the approach that real estate appraisers take when using "comps" to

11. We also illustrate that discounted cash flow (DCF) analysis has many features that at least partially borrow from nearest neighbor matching.

12. *See, e.g.*, Highfields Cap., Ltd. v. AXA Fin., Inc., 939 A.2d 34, 46–47 (Del. Ch. Aug. 17, 2007) (describing expert's use of "three traditional valuation methodologies").

13. *See, e.g.*, *In re* PetSmart, Inc., No. 10782, 2017 WL 2303599, at *24 (Del. Ch. May 26, 2017) ("Metrick asserts that his opinion regarding the fair value of PetSmart at the Merger Price is bolstered by the following confirmatory analyses: (1) his DCF analysis resulting in a value of $81.44 per share; (2) the fact that '[a]t no point prior to PetSmart's acquisition did its shares trade at or above $83 per share'; (3) the fact that '[a]t no point prior to the consummation of the transaction did analysts' average price target of PetSmart exceed $83 per share' . . . .").

estimate the value of one's home. The basic idea is to find examples of analogous assets that have recently been sold in arm's length-transactions, and then to use those sales prices to deliver an estimate of what the sale of the company in question would deliver. With home appraisals, this process usually begins by selecting neighborhoods in a similar geographic area with similar demographic traits, such as walkability, school quality, and income, as well as having a similar number of bedrooms and square footage.[14] The recently sold properties deemed similar to the property in question along these dimensions are an appraiser's comps. The appraiser then typically normalizes the measure of value, such as price per square foot, for each comparable transaction, aggregating the comps using the mean or median of the price-per-square-foot values.[15] This yields a summary measure of the price per square foot to be applied to the property whose value is in question.

The CT approach for companies and other financial assets operates similarly. Much like the property appraiser, a valuation analyst using a CT methodology will first find recent sales of companies that appear to be comparable. If feasible, the analyst will limit attention to transactions in similar industries, geographic locations, or vintages.[16] Like real estate, companies vary in size, so finding comps of similar scale is desirable. In addition, they can have unique capital structure traits: for example, corporate debt often can transfer over as part of the sale, in which case the purchase price reflects only equity value. The valuation analyst will thus attempt to produce a measure of firm value that controls for both equity and debt factors. To account for capital structure, the analyst often needs to rescale the purchase price to reflect what is known as the "enterprise value" of each comparable company, adjusting the sales price of each comp to account for the value of any debt not capitalized into the sales price.[17]

Once comparable sales prices are converted to enterprise values, analysts then address remaining size differences, using an analog of the price per square foot measure used by real estate appraisers. Here, however, the standard normalized metric is typically through an earnings or revenue multiple: That is, re-expressing the firm valuation as a multiple of some specified measure of accounting earnings rather than in raw dollar terms. A standard metric that operates as a default is earnings before interest, taxation, depreciation and amortization (EBITDA), which is often a relatively stable proxy for cash flows in mature companies.[18]

14. SHANNON P. PRATT & ALINA V. NICULITA, VALUING A BUSINESS: THE ANALYSIS AND APPRAISAL OF CLOSELY HELD COMPANIES 321 (5th ed. 2008).

15. JOSHUA ROSENBAUM & JOSHUA PEARL, INVESTMENT BANKING: VALUATION, LEVERAGED BUYOUTS, AND MERGERS & ACQUISITIONS 71–72 (2d ed. 2013).

16. These and other traits are specified in a small set of valuation manuals that have become accepted over time in the profession. *See, e.g.*, PRATT & NICULITA, *supra* note 14; ROSENBAUM & PEARL, *supra* note 15; ASWATH DAMODARAN, INVESTMENT VALUATION: TOOLS AND TECHNIQUES FOR DETERMINING THE VALUE OF ANY ASSET 542–43 (3d ed. 2012) ("The most common approach to estimating PBV ratios for a firm is to choose a group of comparable firms, to calculate the average PBV ratio for this group, and to base the PBV ratio estimate for a firm on this average.").

17. *See, e.g.*, *In re* Appraisal Jarden Corp., No. 12456, 2019 WL 3244085, at *49 (Del. Ch. July 19, 2019) ("In order to determine the final share price under a DCF approach, the appraiser must account for Jarden's excess cash and debt in its enterprise value."). Other adjustments include netting off cash (and cash equivalents), as well as making sometimes controversial changes to working capital. *See, e.g.*, OSI Sys., Inc. v. Instrumentarium Corp., 892 A.2d 1086, 1090–95 (Del. Ch. Mar. 14, 2006) (arguing over which accounting practices control the sale price of a business).

18. Doft & Co. v. Travelocity.com Inc., No. CIV.A. 19734, 2004 WL 1152338, at *10 (Del. Ch. May 20, 2004) ("Gompers agrees with Purcell that the EBITDA multiples are the 'preferred multiple to examine' because they 'are closest to cash flow and are a better proxy for the firm's on-going concern value.'").

For less mature companies, however, it is not uncommon to see multiples based on other factors, such as EBIT, sales revenues, or, less commonly, measures of market interest such as "clicks" on the company's website.[19] Non-EBITDA multiples are typically disfavored,[20] however, and tend only to be used when the company generates negative adjusted earnings, which renders any multiples-based approach nonsensical.

Even after a multiple has been selected, there are many ways to quantify the denominator of the multiple. For example, one might base a multiple on the last fiscal year's numbers, or the last 12 months, or projections for the next 12 months or fiscal year. In each case, the multiple of the comparable company may change, and it is not uncommon for analysts to assess CT multiples using several measures, thereby cobbling together a range of valuations based on the aggregated outcomes of such approaches, as well as a variety of permutations of the mean, median or inter-quartile range for each measure.[21] In formulating the comps, the multiple formulations, and the ranges, the analyst typically retains significant discretion—an issue to which we return in our analysis below.

However, two significant constraints on the CT approach often limit its ability to deliver valuation projections. The first is a lack of data. Because bona fide arms-length sales of companies within a given industry are not an everyday occurrence, the set of comparable recent transactions might itself be relatively sparse, which may force the analyst to make a projection from a very small group (perhaps even as small as one).[22] One potential solution to this problem is to lengthen the time horizon or broaden the criteria by which comparatives are drawn (e.g., by expanding the industries considered).[23] The second potential limitation on the CT approach is that it is predicated on sales of comparable firms through negotiated transactions. Such sales often come with a premium baked into the sales price, reflecting the value of control. This baked-in control premium may sometimes be inappropriate to include in the valuation estimate if (for example) one is interested in gauging only the status-quo / steady state valuation of the company. In such settings, CT requires a further challenging exercise of shaving off any control premia from precedent sales.[24]

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19. Merion Cap., L.P. v. 3M Cogent, Inc., No. 6247, 2013 WL 3793896, at *6 (Del. Ch. July 8, 2013), *enforced*, 2013 WL 3833763 (Del. Ch. 2013) ("The comparable companies method of valuing a company's equity involves several steps including: (1) finding comparable, publicly traded companies that have reviewable financial information; (2) calculating the ratio between the trading price of the stocks of each of those companies and some recognized measure reflecting their income such as revenue, EBIT, or EBITDA; (3) correcting these derived ratios to account for differences, such as in capital structure, between the public companies and the target company being valued; and, finally, (4) applying the average multiple of the comparable companies to the relevant income measurement of the target company, here Cogent.").

20. Our simulation analysis focuses on an EBIT-based measure for data availability reasons (namely negative values of EBITDA). We note that this is not a rare occurrence; in our data sample for the simulation, nearly 30% of firms have negative EBITDA. Thus, we focus on EBIT in scaling earnings to increase sample size in our analyses.

21. DAMODARAN, *supra* note 16, at 458–59.

22. ROBERT F. REILLY & ROBERT P. SCHWEIHS, GUIDE TO INTANGIBLE ASSET VALUATION 185 (2014).

23. BRADFORD CORNELL, CORPORATE VALUATION: TOOLS FOR EFFECTIVE APPRAISAL AND DECISION MAKING 98–99 (1993).

24. Z. CHRISTOPHER MERCER & TRAVIS W. HARMS, BUSINESS VALUATION: AN INTEGRATED THEORY 189–90 (2d ed. 2008).

B. Comparable Companies

The CC approach is a close conceptual cousin to the CT approach, differing principally in the source of the data used to assess comparable firms. Where CT uses data from acquisitions of comparable firms, CC instead uses regularly reported financial data from large, developed securities markets. In such markets, stock prices are thought to be a good proxy for the economic value of a fractional share of the company, at least on average and as viewed by the marginal investor.[25] Thus, rather than using the sales price from previous transactions to predict value, CC is based on comparing public data related to the total market capitalization of comparable firms regardless of whether they have recently been sold. Beyond that difference, CC valuation progresses in a manner similar to CT, including the conversion from an enterprise value multiple to enterprise value; the specification of the multiple itself; the use of judgment about how to measure such multiples, such as last 12 months, next 12 months, etc.; and a summary measure (mean or median) for aggregating comp multiples. In other words, beyond the different source of valuation metrics for the comparable firms, other parts of CC analysis are very similar to CT analysis.

CC methodologies have one obvious advantage over CT approaches: data. It can be difficult to find comparable prior M&A transactions to use in developing CT comps, but CC is facilitated because thousands of public companies trade continuously and have observable prices each day. Consequently, CC allows one to build sizable groups of comparable firms. Because CC also tethers the value of companies to the trading value of their stocks, which in turn tends to reflect the value the market ascribes to a publicly held valuation target, the CC approach neglects the value of the control premium for companies (if one wishes to measure that). That can be problematic when the valuation objective involves a company's value as a private concern. Additionally, when public companies' market value may include a premium due to the liquidity that comes with public listing, making them not directly comparable with private companies. Another potential limitation of the CC approach is that it depends on the on-average value efficiency of trading markets. Such conditions often prevail for sophisticated markets, but CC is more prone to misfiring when the target firm (and/or many of its comps) is traded in a thin, volatile, poorly-developed market where market valuations and fundamental valuations can diverge.

C. Discounted Cash Flow

The third major form of valuation is discounted cash flow (DCF) analysis. Compared to CC and CT approaches, DCF has more moving parts, and it is generally viewed as more technically demanding.[26] Using the DCF approach can be motivated via the comparison of a company's value to the value of holding a financial asset that produces a particular pattern of cash flows over

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25. Verition Partners Master Fund Ltd. v. Aruba Networks, Inc., 210 A.3d 128, 137 (Del. Apr. 16, 2019) ("*Dell*'s references to market efficiency focused on informational efficiency—the idea that markets quickly reflect publicly available information and can be a proxy for fair value . . . .").

26. *See, e.g.*, Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd., 177 A.3d 1, 37–38 (Del. Dec. 14, 2017) ("Although widely considered the best tool for valuing companies when there is no credible market information and no market check, DCF valuations involve many inputs—all subject to disagreement by well-compensated and highly credentialed experts—and even slight differences in these inputs can produce large valuation gaps. Here, management's projections alone involved more than 1100 inputs, and the experts' fair value determinations (which also included several novel tax issues discussed below) landed on different planets.").

time: It is the equivalent to the present discounted value of the free cash flows the company is projected to generate over a given time horizon. Borrowing from the well-known formula in finance for present values, the DCF approach[27] can be captured as follows:

$$FMV = PV(Cash\ Flows) = \frac{FCF_1}{(1+WACC)} + \frac{FCF_2}{(1+WACC)^2} + \cdots + \frac{FCF_T}{(1+WACC)^T} + \frac{S_T}{(1+WACC)^T}$$

where $t = 1, 2, 3, \ldots T$ indexes time period, with period T being the terminal period; FCF_t denotes expected future "free cash flows" in period t; S_t denotes a terminal (or "salvage") valuation of the asset as of the terminal projection year; and WACC represents a risk-adjusted discount rate known as the "Weighted Average Cost of Capital." If all of these ingredients are known or can be reliably estimated, then a cash-flow valuation is possible.

Like an onion, each of the ingredients of the DCF approach has its own layers of complexity (and resulting expert discretion). Free cash flows are sometimes generated from analyst forecasts, or through management forecasts done in the ordinary course, or through investment banker forecasts for the purposes of a (disputed) transaction, or by some other source.[28] They are typically, though not always, "unlevered," meaning they include interest payable to capital creditors, so as to summarize the entire pool of earnings available to satisfy both debt and equity holders.[29] And, in some cases, cash flow projections of comparable firms (if available) can be used to form composite projections that more closely track the industry at large.[30]

The WACC discount rate is typically a blend of expected return estimates for debt and equity (adjusted for leverage ratios and tax deductions), with equity return estimates being the product of an underlying asset pricing model (such as the still-dominant market "beta" from the capital asset pricing model).[31] In many cases, peer company information is also blended in with company-specific data to create more of a composite measure (usually after an elaborate process adjusting peers' betas for differences in leverage ratios).[32]

Finally, the terminal value measure (S_T) accounts for the fact that available projections of free cash flow do not stretch indefinitely into the future. Because company projections are typically no longer than ten years (and are more frequently five to seven years), an analyst using DCF must make an assumption of what the asset will be worth as of its "terminal" period (when no more forward-looking projections are available). The assessment of terminal value is perhaps the most susceptible to expert degrees of freedom since there is little to tether it to firm-level data. One approach for terminal values is to extrapolate the final period's free cash flow indefinitely into the future as a "growing perpetuity" at some posited growth rate, backing out a present valuation (as of period *T*) from a well-known formula for the present value of

27. *See, e.g.*, *In re* Vanderveer Ests. Holding, LLC, 293 B.R. 560, 578 (Bankr. E.D.N.Y. May 8, 2003).

28. TIM KOLLER, MARC GOEDHART & DAVID WESSELS, VALUATION: MEASURING AND MANAGING THE VALUE OF COMPANIES 230–31 (7th ed. 2020).

29. DAMODARAN, *supra* note 16, at 285–86.

30. SHANNON P. PRATT & ROGER J. GRABOWSKI, COST OF CAPITAL: APPLICATIONS AND EXAMPLES 583–84 (5th ed. 2014).

31. RICHARD A. BREALEY, STEWART C. MYERS & FRANKLIN ALLEN, PRINCIPLES OF CORPORATE FINANCE 214–15 (13th ed. 2020).

32. *See, e.g.*, Ramcell, Inc. v. Alltel Corp., No. 2019-0601, 2022 WL 16549259, at *19 (Del. Ch. Oct. 31, 2022).

growing perpetuities.[33] Another frequently used approach, however, is simply to revert (once again) to peer company multiples using a recycled CC or CT approach applied as of the terminal period.[34]

III. The Conventional Approaches and Nearest Neighbors Matching Algorithms

In Section II, we described conventional approaches to valuing firms for litigation purposes. We focus on two key points from that analysis here. First, expert discretion in valuing companies ranges from the choice of approach (e.g., CC, CT, or DCF) to how to measure inputs used in that approach. Second, peer-group comparisons are pervasive—indeed, such comparisons are the very backbone of CC and CT analysis, and even in DCF analysis, peer group comparisons may play a role in generating the expert's free cash flow projections, terminal values, and beta estimates.

When considering the scope of the expert's task, it is noteworthy how closely valuation practice resembles what statisticians and data scientists refer to as the *k*-nearest neighbor (*k*-NN) algorithm from machine learning. First, valuation experts' goal can be viewed as making predictions ("what would the value of this firm be under particular circumstances"), rather than as testing causal theories. Second, valuation experts use a significant amount of unstructured data to deliver those predictions. And third, valuation experts habitually use putatively similar examples in the population—styled as "comps"—in executing the predictive enterprise.

These approaches may be viewed as instances of the *k*-nearest neighbor (*k*-NN) algorithm. This algorithm, which was first developed over a half century ago,[35] is a non-parametric approach to statistical learning[36] that is often used to classify or predict a target variable of interest. The parameter k represents the number of closest neighbors/comps of the target that are to be used to create the prediction. When multiple characteristics (often called features in the machine-learning literature) of firms are observable and available for use in value prediction, the set of neighbors that are closest to the target will depend on how experts choose to measure distance between firms, as we discuss in more detail below. The *k*-NN method can be used for both classification problems—where the object is to predict which of a discrete set of categories the item of interest belongs[37]—or continuous problems, where the objective is to predict the value of a continuous outcome variable such as firm value.

In the simplest *k*-NN setting, only one variable, x, is used in matching, and the *k*-nearest neighbors are the k firms with the closest values of x to the target unit. These k firms are thus the valuation expert's comps. The predicted firm value for the target firm is typically taken to be the average of the k comps' values (the median of the comps' value also

33. *See, e.g.*, *In re* Vanderveer Ests. Holding, LLC, 293 B.R. 560, 578 (Bankr. E.D.N.Y. May 8, 2003).

34. *Ramcell*, 2022 WL 16549259, at *19.

35. *See, e.g.*, Evelyn Fix & Joseph L. Hodges, Discriminatory Analysis: Nonparametric Discrimination, Consistency Properties (1951); *see also* Thomas M. Cover & Peter E. Hart, *Nearest Neighbor Pattern Classification*, 13 IEEE Transactions on Info. Theory 21 (1967).

36. Statistical learning is the attempt to "extract important patterns and trends, and understand 'what the data says.'" Trevor Hastie, Robert Tibshirani & Jerome H. Friedman, The Elements of Statistical Learning: Data Mining, Inference, and Prediction xi (2d ed. 2009).

37. For example, is the image a cat or a dog?

is sometimes used). The *k*-NN algorithm can be modified to allow more than one *x*-variable. Speaking generally, its key ingredients are:

(a) an outcome variable y, whose value (often called a "label" in machine learning communities) is to be predicted;

(b) a vector of characteristics, X, whose values will be used to assess the distance between the target and its neighbors;

(c) a distance metric, which is a function that, when applied to any pair of X-vector values, measures the distance between them (one example of a distance metric is Euclidean distance);

(d) a rule for selecting which neighbors will be used to predict the target unit's value of y (e.g., "use the 3 closest neighbors to the target firm based on the distance metric in step (c)"); and

(e) a function that combines the values of the selected neighbors' values, $\{y_i\}_{i=1}^{k}$, into a single prediction, $\hat{y}$ (e.g., the mean or median of the selected neighbors' values).

This discussion reveals that the Comparable Companies and Comparable Transactions valuation methods are not just *similar* to the *k*-NN algorithm—they are in effect *instances* of the *k*-NN algorithm. Indeed, both the CC and CT approaches involve the following steps:

(a) using an earnings multiple as the outcome variable y (for CC, y is trading value, and for CT it is acquisition value);

(b) using a set of firm variables as the X data used to identify comparable companies or transactions;

(c) & (d) deciding how similar peer firms are to the target firm and which of these comps should be used to predict the target firm's earnings multiple (in practice these steps are rarely specified precisely); and

(e) using some aggregation method—such as the mean or the median—to predict the target firm's earnings multiple.

To the best of our knowledge, neither the CC method nor the CT method has previously been viewed as an instance of the *k*-NN algorithm, but it is clear from the discussion above that this identification is accurate.

The DCF valuation method does not map as neatly into the *k*-NN algorithm, but it still shares some important characteristics. For example, it is common in DCF practice to use comps to generate estimates of terminal value, of earnings projections, and of asset betas—all three of which are core ingredients of DCF valuation methods. Thus, *k*-NN's spirit, if not always its letter, typically enters DCF analysis at several junctures.

Given that standard valuation methodologies are, or are similar to, *k*-NN learners, we can understand their advantages and disadvantages by reference to those of *k*-NNs. The *k*-NN method is a form of instance-based learning, which means that, unlike other supervised learning approaches, it need not require a training stage for use.[38] That makes *k*-NN simpler and faster to use than many alternative algorithms. And as the data set grows, *k*-NN converges to the Bayes optimal error rate.[39]

On the other hand, *k*-NN has several limitations that can hamper its performance. First, as the amount of data grows, computational costs increase very rapidly, because pairwise similarities must be calculated for all plausible comps.[40] *k*-NN use is also complicated by the potentially large number of X variables, which can lead to the so-called "curse of dimensionality."[41] Third, *k*-NN can be highly sensitive to scale and how distance is calculated. Good performance using *k*-NN often requires normalization of variables before applying the algorithm—which adds another layer of expert discretion, because multiple normalizing scalers exist.[42] Finally, *k*-NN can be sensitive to noisy data, missing values, and outliers.[43]

This discussion indicates that actual *k*-NN performance will depend on a number of factors whose net contributions are hard to characterize *a priori*, and which entail expert discretion. To see how these factors interact, we turn next to a series of data-based simulations to assess the performance of *k*-NN, as well as other data-driven approaches, using real firm data.

IV. Monte Carlo Simulations Using k-NN

In this section, we conduct a series of Monte Carlo simulations using actual firm data to assess the performance of *k*-NN and alternative data-driven valuation approaches. To facilitate the illustration, we limit our analysis to comparable companies (CC) analysis, and we simulate the behavior of motivated experts, who are assumed to use CC to generate either a high (but plausible) valuation in their expert opinion, or a low (but plausible) one, respectively.[44]

Because CC utilizes comparable data from public companies, we draw on such data here. Our primary data source consists of the Compustat quarterly financial data from 2000

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38. David W. Aha, Dennis Kibler & Marc K. Albert, *Instance-Based Learning Algorithms*, 6 Mach. Learning 37, 41–43 (1991). That said, it may be desirable to train a *k*-NN learner, to the extent one is unsure how to set *k*, what distance/similarity measure to use, how to aggregate the outcome label for comps, and so forth. We consider these issues below.

39. *See* Hastie, Tibshirani & Friedman, *supra* note 36, at 465 ("[A]symptotically the error rate of the 1-nearest-neighbor classifier is never more than twice the error rate.").

40. *Id.* at 480. This is unlikely to be a serious problem in firm valuation problems, so long as the set of potential comps can be limited on preliminary grounds, e.g., by focusing on firms that share common industries of operation.

41. Gareth James et al., An Introduction to Statistical Learning with Applications in R 106–07, 161 (2013).

42. Gareth James et al., An Introduction to Statistical Learning: With Applications in R 39–42 (2d ed. 2021).

43. Hastie, Tibshirani & Friedman, *supra* note 36, at 463–64.

44. Alternatively, a litigant could shop for multiple experts and submit to the court only the most favorable report. For discussions of such "expert mining," see generally Jonah B. Gelbach, *Expert Mining and Required Disclosure*, 81 U. Chi. L. Rev. 131 (2014); *see also* Christopher T. Robertson, *Blind Expertise*, 85 N.Y.U. L. Rev. 174 (2010).

to 2020 for all public reporting companies in the United States with a fiscal-year reporting date of December 31.[45] We merge this data with stock price data from the Center for Research in Security Prices (CRSP), keeping only verified links.[46] Because stock prices change continuously, we match the prevailing quarterly information from Compustat to daily individual security prices and index returns in CRSP; we additionally also match the data to data on the well-known Fama-French-Carhart factors that are provided on Ken French's website (and are a workhorse for contemporary asset pricing).[47]

We next developed lists of matching characteristics for choosing company peers based on the discussion in two leading corporate finance textbooks that supply potential best practices in choosing comps.[48] We drop any observation for which any of the characteristics recommended by either textbook is missing.[49]

In addition, we categorize each firm-quarter observation as belonging to an industry, defined by the first two digits of the firm's Standard Industrial Classification (SIC) code.[50] We impute missing industry designations using a "down-up" strategy, which assumes (i) that all missing values before the first recorded non-missing entry are equal to the first entry, and (ii) that subsequent missing values are equal to the most recent entry until a new entry is recorded.[51] We drop all observations for firms in the financial services industry, because their capital structure and investment policies differ significantly from those in other industries.

Having constructed a panel dataset of firm-quarter observations, we then randomly sample 10,000 firm-quarter observations for use as what we call *firm-date targets* in our simulations. In order to be selected as a firm-date target observation, we require firm-quarter observations to have (i) at least nine peer firms that have full non-missing data in the selected quarter, and (ii) at least eight consecutive non-missing values, ending on the selected quarter, for the ratio of market capitalization to EBIT (earnings before interest and taxes). In addition, for each firm-date target observation, we choose a simulated valuation date by selecting a random number of trading days, d^*, between zero and fifty, after the end of the quarter to value the firm. For example, if the randomly selected observation is for the third fiscal quarter of 2015, which ends on September 30, 2015, and we draw $d^* =$

45. We restrict our attention to 12/31-reporting firms so that the different fiscal-year quarter ends align in calendar time. In additional results that are available on request, we find that this restriction does not drive our results.

46. These are link codes equal to "LC", "LU", or "LS".

47. *See Current Research Returns*, KENNETH R. FRENCH, https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html [https://perma.cc/KPB6-69F4].

48. PRATT & NICULITA, *supra* note 14, at 321; ROSENBAUM & PEARL, *supra* note 15, at 71–72. We provide a list of firm-quarter variables, and their calculation method and data source in the Data Appendix.

49. To be clear, we do not necessarily drop *firms* in this case, but rather just the observations that have missing values. For example, if there are 16 dates with observations on firm *f*, of which 10 observations have non-missing data and 6 have some missing data, we would drop the 6 observations that have missing data but still include the 10 observations with all non-missing data.

50. We employ Compustat's historical identifier, *sich*, which allows for time variation in industry designation. For our purposes, it is important to use *sich* rather than the commonly used *sic* variable, because the value of the latter is the most recently identified industry code for all observations (which can change over time). Consider a firm that shifts from industry A to industry B at time $t - 1$. If Compustat data is downloaded at time t, the value of the *sic* variable will be industry B's code for all time periods (even though the firm switched industries).

51. If *sich* has a missing value for the last *S* observations, we use the value of *sic*.

45, then the target valuation date is the 45th trading date following the third-quarter's end, which is December 3, 2015.

For each of our 10,000 firm-date targets, we are able to compute the following:

- The actually observed *firm valuation* (understood as market capitalization) on d^*;
- The actually observed *valuation ratio* (the ratio of market capitalization to EBIT);
- Lagged values of the firm valuation and valuation ratio (i.e., values for the period before the valuation target date);
- An industry categorization;
- A set of covariate values typically used by practitioners for identifying comps (hereinafter "*X*" variables).

Our simulation entails using various prediction methods for each of the 10,000 firm-date targets. Each method yields a prediction value for the firm-date target valuation and valuation ratio variables for each prediction method. However, because we conduct our simulations using actual data, we also know the firm's observed value on the given date. For each prediction method, then, we calculate the prediction error as the difference between the *predicted* and the *observed* values of the target variables. Finally, we assess the performance of each method by comparing statistics related to these prediction errors.

To illustrate our approach, consider an example of predicting the value of a randomly-sampled firm on a randomly-sampled date target. Table 1 shows basic financial information about the firm in question, which is the NASDAQ-listed transportation services company Landstar System, Inc. The target valuation quarter is the 4th quarter of 2017; with $d^* = 43$. The target valuation date is the 43rd trading day of 2018, which was March 5, 2018.[52] The firm is in industry code 42, representing motor-freight transportation.[53] Landstar's market capitalization was $4,371 million and its EBIT was $70, so that the valuation ratio—market cap divided by EBIT—was 62.40.

Table 1: Motivating Example: Landstar System

Company	Quarter	Industry	Market Cap (M)	EBIT	Ratio	d^*
LANDSTAR SYSTEM INC	2017Q4	42	$4,371	$70	62.40	43

1. Conventional k-NN Approach and Motivated Experts

In Part III, we described how the conventional approach to valuation described in leading corporate finance textbooks represents an instance of *k*-NN matching. In this

52. *2026 Trading Days Calendar,* SWINGTRADESYSTEMS.COM, https://www.swingtradesystems.com/trading-days-calendars.html [https://perma.cc/82DY-YJFL] (using the "2018 Trading Days Calendar").

53. *Major Group 42: Motor Freight Transportation and Warehousing*, OSHA, https://www.osha.gov/data/sic-manual/major-group-42 [https://perma.cc/E49J-VGYU].

section, we use our simulation approach to test the performance of the conventional approach, and along with it, its susceptibility to expert discretion.[54]

Our Monte Carlo simulation proceeds as follows. We proceed much like the single example of Landstar System above; we select 10,000 examples of random public companies on a random target date (which must fall within a single quarter). For each target firm j and implied target quarter q, we denote the target-quarter valuation ratio r_{jq},[55] the firm's market capitalization on the target date d^* as mc_{jd^*}, and the firm's valuation ratio on that date as $r_{jq}^{d^*}$.[56] Our prediction exercises below will (by turns) highlight these three market cap and valuation ratio variables.

Having defined target variables, we next identify all viable peer firms for a target firm and quarter. For each firm, we limit consideration to other firms that (i) are in the same two-digit SIC code industry, (ii) have non-missing entries for all posited covariates at relevant times, and (iii) have full market trading data for the 250 trading days prior to quarter's end plus the d* days following the quarter. By way of example, Table 2 lists the twelve potential peer firms that would satisfy these three criteria for Landstar System, with target quarter 2017 Q4 and $d^* = 43$.

Table 2: Potential Peer Firms Within Same Industry (Landstar System example)

HUNT J B TRANSPORT SERVICES INC	FORWARD AIR CORP
WERNER ENTERPRISES INC	KNIGHT SWIFT TRANSPORT HDLGS INC
P A M TRANSPORTATION SVCS INC	COVENANT TRANSPORTATION GRP INC
MARTEN TRANSPORT LTD	CH ROBINSON WORLDWIDE INC
HEARTLAND EXPRESS INC	SAIA INC
OLD DOMINION FREIGHT LINE INC	UNIVERSAL LOGISTICS HOLDINGS INC

This table reports the potential peer firms for our motivating example. Landstar System Inc. had a two-digit SIC code of 42 in Q4 2017, which is Motor Freight Transportation and Warehousing. The 12 peer firms in this table are those with the requisite data during the relevant time period.

Next, we calculate predictions for each target firm-date using k-NN. As previously discussed, there are multiple discretion-according choices involved. Here we focus on four: (i) which X variables to use to assess firms' similarity (or "distances") to one another; (ii) how many neighbors to use (i.e., the value of k); (iii) how to measure distance; and (iv) whether to aggregate comp firms' valuation variables using the mean or the median.

54. To be clear, we are not arguing that our k-NN implementation exactly replicates *how experts produce* valuation estimates under the conventional approach. It should instead be seen as a systematic implementation of the textbook approach that leaves open numerous areas of discretion. By clarifying the design considerations in the matching algorithm, our k-NN approach is, if anything, likely to have less discretion than implementing the CC method as the latter is used in litigation. Inasmuch as our point is to demonstrate that there is a large amount of discretion, our approach is thus methodologically conservative.

55. This is the ratio of firm j's end-of-quarter market cap, mc_{jq}, to its end-of-quarter EBIT, $EBIT_{jq}$, so that $r_{jq} = \frac{mc_{jq}}{EBIT_{jq}}$.

56. Thus, $r_{jq}^{d^*} = \frac{mc_{jd^*}}{EBIT_{jq}}$; notice that we use the daily value of market cap for date d^*, together with the value of EBIT for the preceding quarter, q; the notation for the ratio accounts for both these indices.

We illustrate how this works using our Landstar example. Table 3 lists the five nearest and farthest potential Landstar peers (among those that were listed in Table 2) using the matching variables specified in the Rosenbaum & Pearl corporate finance textbook. The distance between Landstar System and each potential peer firm was calculated using the scaled Euclidean distance metric. Using k=5, our comps would be Hunt (JB) Transport Services, Werner Enterprises, P.A.M. Transportation Services, Martin Transport, and Heartland Express. Our final step would be to aggregate the values of these firms. If our target variable is the ratio of market cap to EBIT, $r_{jq}^{d^*}$, then we would use 77.55, because this is the median value of the Ratio column in Table 3 for the five comps. That is, $r_{jq}^{d^*} = 77.55$. Because this is the predicted ratio of market cap on date d^* to EBIT at the end of the most recent quarter before d^*, the comps-based predicted value of Landstar System's market cap on date d^* may be found as 77.55 times the firm's value of $EBIT_{jq}$. The EBIT value for Landstar Systems at the end of 2017 Q4 was $70 million.[57] Therefore, our predicted market cap for Landstar System is $\widehat{mc}_{jd^*} = r_{jq}^{d^*} \times EBIT_{jq} = 77.55 \times \$70M = 5{,}429M$, or roughly $5.4 billion.

Table 3: Best and Worst
Nearest Neighbor Matches for Landstar System Motivating Example

Company	EBIT	Market Cap (M)	Ratio	Rank
Best Matches				
HUNT (JB) TRANSPRT SVCS INC	166.17	$12,886	77.55	1
WERNER ENTERPRISES INC	44.29	$2,706	61.09	2
P.A.M. TRANSPORTATION SVCS	3.03	$225	74.12	3
MARTEN TRANSPORT LTD	13.72	$1,187	86.53	4
HEARTLAND EXPRESS INC	3.04	$1,625	534.68	5
Worst Matches				
KNIGHT-SWIFT TRPTN HLDGS INC	137.40	$8,511	61.94	8
COVENANT LOGISTICS GROUP INC	15.65	$413	26.42	9
C H ROBINSON WORLDWIDE INC	210.88	$12,600	59.75	10
SAIA INC	23.22	$1,810	77.97	11
UNIVERSAL LOGISTICS HLDGS	13.11	$607	46.32	12

This table reports the five best and five worst matches for our motivating example using a *k*-nearest neighbors matching approach. We use the covariates from Rosenbaum and Pearl (RP), and a scaled Euclidean distance metric.

Even given that we targeted the valuation ratio, there were four dimensions of discretion in the approach just described: the set of matching variables (Table 3 uses the Rosenbaum & Pearl set), the number of peers used as comps (our discussion of Table 3 uses 5), the distance metric (Table 3 uses the scaled Euclidean metric), and the aggregation measure (our discussion uses the median). In practice, an expert has discretion over these choices. To explore the scope of this discretion, we predict firm value using 24 different combinations of *k*-NN parameter choices, with each representing a distinct combination of choices from the following alternatives:

57. *See supra* Table 1 (representing the EBIT value of Landstar).

- Two covariate sets (those described in Rosenbaum & Pearl (RP), or those described in Pratt & Niculita (PN))[58];
- Three values of k (5, 7, or 9);
- Two distance metrics (scaled Euclidean (SE) or Mahalanobis (M));
- Two ways of aggregating comparable valuation ratios (mean or median).

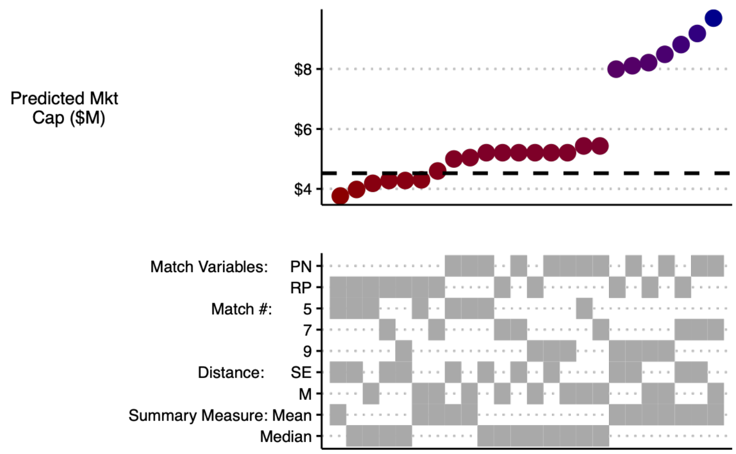
For each of the 24 possible combinations of these four types of parameters, we calculate a predicted valuation ratio, $\hat{r}_{jq}^{d^*}$, for firm j and date d^*, just as we did in discussing the Landstar System example in Table 3 above. We then multiply that predicted value by the actual $EBIT_{jq}$ for target firm j and quarter q. The resulting product is our market cap prediction, $\widehat{mc}_{jd^*}$, for the particular combination. As noted, there are 24 such values for each firm.

Figure 1 reports these predicted valuations for Landstar System. The top panel displays the estimates in order from smallest to largest. The range is remarkably wide: by choosing among the 24 options, motivated experts could generate estimates as low as $3.8 billion, and as high as $9.7 billion. The observed valuation on date d^*, represented by the dashed line in the top panel, was $4.5 billion.

The figure's bottom panel displays the specific parameterization for each estimate, with shaded gray cells indicating the values of the four inputs used. For example, the lowest predicted value ($3.8 billion) occurs when the RP covariate set is used, $k = 5$ peer firms are used as comps, the scaled Euclidean distance metric is used, and comps' valuation ratios are aggregated using the mean. The bottom panel shows that there is no obvious regularity linking parameterization choice to predicted valuation magnitude; each value of each of the four inputs is associated with both high and low predicted market cap values.

58. A list of these matching variables and their construction is presented in the Data Appendix.

Figure 1: Input Choice and Valuation
(Landstar System example)



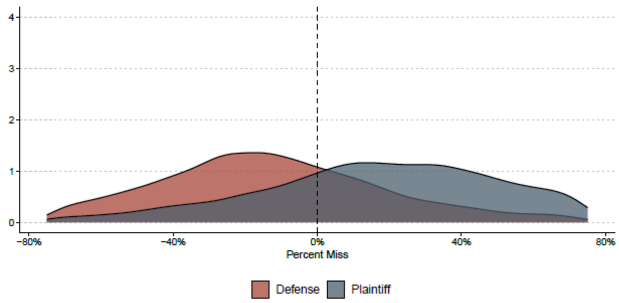
This figure reports the valuation estimates for Landstar $d^* = 43$ trading days after the end of 2017 Q4 using different input combinations. The top panel plots the estimated market value from each of the 24 combinations, which are represented by the grey tiles in the lower panel. The combinations vary based on the choice of matching feature set, number of matches, distance measure, and summary measure.

Figure 1 thus illustrates the great degree of expert discretion that conventional valuation approaches afford. We believe this fact helps explain why expert valuation reports in litigation typically vary so widely between defense and plaintiff experts: Even restricting consideration to approaches that are squarely within textbook accounts of best practice, it is possible to find an enormous range of variation in valuation amounts.

Of course, that analysis was for just one firm-date target, and perhaps there is something especially unrepresentative going on for Landstar System in the 4th quarter of 2017. To address this possibility, we calculate the same 24 predicted valuations for each of 9999 *additional,* randomly-selected firm-date targets. As with Landstar, for each of these additional target-date observations, we calculate the predicted valuations for each of the 24 input permutations (two choices of matching variables, three choices of matching number k, two different distance measures, and two summary measures). We then assume that a defense expert includes the three lowest estimates in her report, favoring the middle estimate (the second-lowest overall) in order to appear moderate. We assume the plaintiff expert does the same, reporting on the three highest estimates and endorsing the middle one (which is the second highest overall).[59]

59. An alternative way to interpret the strategy would be to assume that the litigation team surveys experts and selects the second lowest or highest estimate; *see* Gelbach, *supra* note 44, at 143–44 (stating a litigant may engage in expert shopping by retaining multiple valuation experts and presenting only the most favorable

Figure 2: Distributions of Hypothetical Defense-Expert and Plaintiff-Expert Predicted Valuations



This figure reports the distribution of the second-lowest and second-highest valuation estimates from the permutations of the conventional k-NN approach, measured by the percentage deviation from the observed market capitalization on the trading date d^* periods after the end of the fiscal quarter. We assume that the defense expert picks the second lowest valuation, and the plaintiff expert selects the second highest.

Figure 2 plots kernel density estimates[60] for the percentage deviation between the hypothetical defense and plaintiff valuations and the observed firm value.[61] To keep outliers from distorting this picture, we restrict the graph to estimates that are within 75% of the observed value. The estimates separate into two visually distinct distributions, whose modes (i.e. the highest-density point) occur at roughly 25% (hypothetical plaintiffs) and -25% (hypothetical defendants). This yields the striking conclusion that even *within k*-NN methodology, the practical variation in valuation attributable to expert degrees of freedom is roughly half the value of the target firm. That suggests that valuation disparities commonly mentioned by the judiciary[62] could simply be driven by experts using the discretion afforded by the conventional *k*-NN approach.

One potential remedy to this problem would be to use an aggregate measure of the predicted valuations. We do this in Figure 3. In Panel A, the gray-shaded curve is the kernel

opinion to the court). Given the cost of producing expert analyses, this seems like the less plausible thought experiment.

60. Kernel density estimation is a method to estimate, non-parametrically, the probability density function of a variable. In essence, it functions as a smoothed histogram. *Kernel*, WIKIPEDIA (June 29, 2026), https://en.wikipedia.org/wiki/Kernel [https://perma.cc/97GP-DTQV]

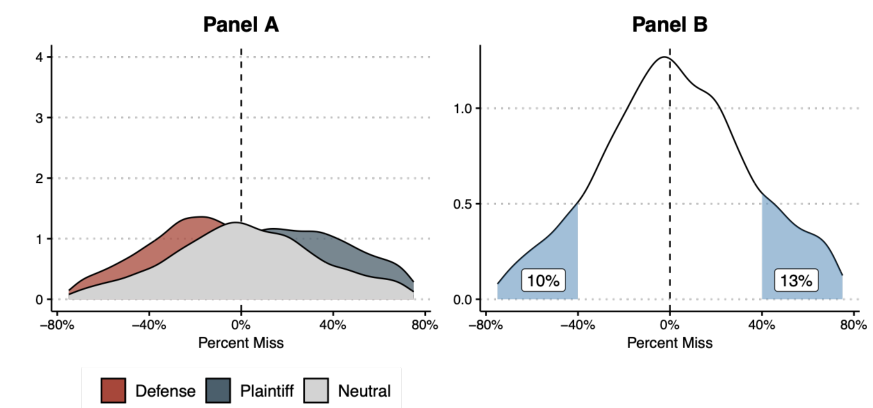
61. We focus on percentage deviations because not all predictions are on the same scale (scale differences are particularly likely to occur because the median aggregation measure is robust to outliers, whereas the mean is not; consider that the highest valuation ratio among the 5-nearest comps in Table 3 is over 500, whereas the median is 77.55).

62. *In re* Appraisal Dell Inc., No. 9322, 2016 WL 3186538, at *45 (Del. Ch. May 31, 2016), *aff'd* i*n part, rev'd in part sub nom.*, Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd., 177 A.3d 1 (Del. 2017) ("Two highly distinguished scholars of valuation science, applying similar valuation principles, thus generated opinions that differed by 126%, or approximately $28 billion. This is a recurring problem.").

density for the distribution of a "neutral" set of estimates for the target firm date. We envision that a judge could order the experts (or independent staff) to build up a combination of all 24 alternative *k*-NN valuations. For example, if both experts used 7 comparable firms, the judge will only consider combinations with 7 comps. But if the defendant's expert favored 7 comps while the plaintiff's expert used only 3, then the judge would order valuations for those two valuations and all in between (here this would imply generating separate analysis of the closest *k* firms, with *k* between 3 and 7. Similarly, if both experts use the same distance metric, then the judge would not require additional analysis using any other distance metric, whereas if they use different distance metrics then the judge would order contingent analyses under both approaches. We then assume that the judge would simply choose the median valuation in the set of "connected dot" combinations that the judge has requested.

For comparison's sake, we superimpose this median curve over the curves for the hypothetical plaintiff and defense estimates from Figure 2, and we provide a dashed line at the 0% deviation value. The figure shows that the target firm-date neutral estimates are centered at this 0%-deviations line, indicating a particular type of accuracy of using the median of considered *k*-NN parameterizations. However, there is substantial variation over the 10,000 target firm-dates, indicating that these neutral estimates are at best a noisily accurate measure; the density appears to have distributional tails analogous to those of the two competing experts, reinforcing the conclusion that noise pervades the enterprise.

Figure 3: The Distribution of Permutation Medians



This figure reports the distribution of the neutral valuation estimate from the permutations of the conventional *k*-NN approach, measured by the percentage deviation from the observed market capitalization on the trading date $d*$ periods after the end of the fiscal quarter. In Panel A we report this median estimate over the plaintiff and defense density estimates. In Panel B, we report the mass of the distribution falling outside of $\pm 40\%$ of the observed market capitalization.

Panel B of Figure 3 isolates attention on the kernel density estimate for the median of the considered combinations of inputs to *k*-NN (note the different scales in Panels A and

B). It shows that 10% of the predictions have a "percent miss" value of exceeding 40% below the observed value, and 13% of the predictions are greater than 40% above.[63]

2. Data-Driven Approaches

As we have seen, the conventional approach to valuation provides substantial discretion for experts and produces estimates with large variance. In this section, we explore whether more contemporary data-driven approaches to prediction can reduce discretion and variance.[64] We show that not only can such approaches improve on the status quo, but they can do so substantially.

1. Using Time-Series Information

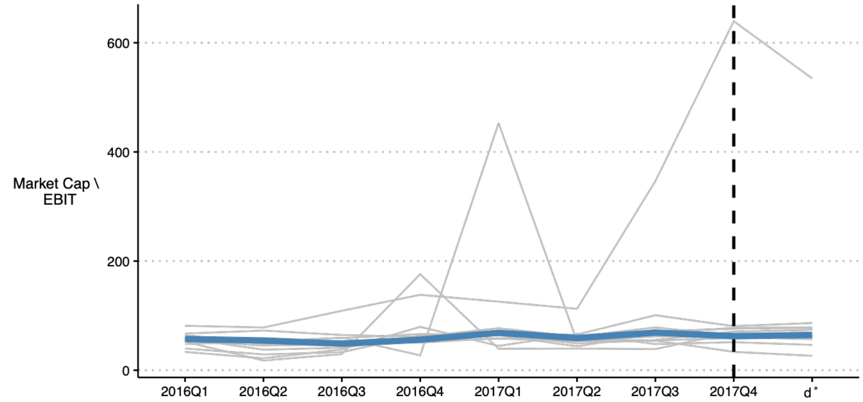
Our first refinement to conventional practice involves using time series data on valuation ratios that the k-NN approach ignores.

Consider again our motivating example of the target firm Landstar System, with target quarter 2017Q4. Figure 4 plots the time series of Landstar System's valuation ratio, $r_{jq}^{d^*}$, for the 8 fiscal quarters before the target quarter (here, $d^* = 0$, i.e., the ratios are calculated with market capitalization on the last day of quarter q). Landstar System's valuation ratio time series is plotted in blue, and the valuation ratio time series for other firms in the same industry are plotted in gray. Landstar's ratio hovers around 60 throughout this period. Several of the firm's peers have ratios that are similar in magnitude, with a couple of peers exhibiting much larger (and much more variable) ratios. This fact likely explains why the densities from the conventional k-NN approach in Figure 1 have such fat right tails (especially when focusing on the mean of the valuation ratios).

63. If we consider the full range of estimates, i.e. even those outside of ± 75% of the observed value, these numbers are even larger, especially for the distribution's very fat right tail.

64. As noted above, neither the conventional nor the data-driven approaches to valuation account for the control premium. This reflects the way valuation has been conducted in appraisal actions under DGCL § 262, though it may not be as appropriate for other areas of litigation that use valuation. However, to the extent that the control premium needs to be modeled separately based on the economics of the firm or industry that is subject to the dispute, it can be added to more a more precisely-estimated walk-away value.

Figure 4: Time Series Dynamics of Valuation Ratio (Landstar System Example)



This figure reports the quarterly valuation ratio (market capitalization over EBIT) measures for Landstar and its peer comparables. The time series for Landstar is represented in blue while the peers are represented in gray. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

The valuation ratio time series properties in Figure 4 suggest a straightforward method for predicting Landstar's target-date valuation ratio: rather than using the kinds of cross-sectional covariates that professional guides suggest for generating comps, we can instead simplify things by concentrating on the time series of valuation ratios for Landstar and its peers before the target date to predict the company's target-date value. In doing this, we use additional time series data on valuation ratios for dates before the target date; such data are readily available in comparable companies analysis situations. In turn, that allows us to eliminate the need (1) to select financial characteristics for use in *k*-NN analysis, (2) to choose measures of distance between comps, and (3) to choose a method for aggregating comps.

To predict the target-date valuation for our target firm, we use valuation-ratio data for the eight quarters before the target date to train a model for firm valuation. These valuation ratios are the only feature we will use for predicting Landstar's value; the model's estimates effectively tell us what weight each peer should receive, yielding an approach that is very similar to so-called synthetic controls.[65]

65. *See* Alberto Abadie, Alexis Diamond & Jens Hainmueller, *Synthetic Control Methods for Comparative Case Studies: Estimating the Effect of California's Tobacco Control Program*, 105 J. AM. STAT. ASS'N. 493–505 (2010); Nikolay Doudchenko & Guido W. Imbens, *Balancing, Regression, Difference-In-Differences and Synthetic Control Methods: A Synthesis* (Nat'l Bureau of Econ. Rsch., Working Paper No. 22791, 2016). Unlike some earlier papers in this literature, though, we implement this idea using three flavors of penalized regression—lasso, ridge, and elastic net. Each of these can be understood as the estimator $\hat{\beta}$ that solves the problem

$$\hat{\beta} = \arg\min_{\beta} \sum_t (y_t - X_t'\beta)^2 + \lambda\left(\alpha \sum_h |\beta_h| + \left(\frac{1-\alpha}{2}\right) \sum_h \beta_h^2\right),$$

for some dependent variable y, where t indexes observations and h indexes the covariates in X_t and the coefficients β_h that multiply them. The scalars α and λ are tuning parameters. When $= 0$, $\hat{\beta}$ is the ridge estimator, for

Table 4: Penalized Regression Weights on Peer Firms (Landstar System Example)

Company	Lasso	Ridge	Elastic Net
INTERCEPT	-14.9506	40.7792	-21.8043
C H ROBINSON WORLDWIDE INC	0.1201	0.0847	0.3367
COVENANT LOGISTICS GROUP INC	0.0000	0.0022	0.0000
FORWARD AIR CORP	0.0000	0.0497	0.0000
HEARTLAND EXPRESS INC	-0.0150	0.0009	-0.0090
HUNT (JB) TRANSPRT SVCS INC	0.7620	0.0479	0.5195
MARTEN TRANSPORT LTD	0.0000	0.0216	0.0000
OLD DOMINION FREIGHT	0.0000	0.0252	0.0000
P.A.M. TRANSPORTATION SVCS	-0.0013	-0.0017	-0.0097
SAIA INC	0.0000	0.0200	0.0000
UNIVERSAL LOGISTICS HLDGS	0.0038	0.0033	0.0495
WERNER ENTERPRISES INC	0.3529	0.0589	0.4605

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is the ratio of market capitalization to EBIT for the target firm, and the features that enter the regression are the ratios for the peer firms. We use quarterly data for the preceding two years in fitting the model and optimize the tuning parameter using leave-one-out cross validation.

Table 4 reports the estimated coefficients associated with these three penalized regression approaches.[66] The lasso specification yields coefficients of 0 for five of the industry peers, with nonzero coefficients resulting for six others. Interestingly, two of these are negative, though small, which indicates that the model predicts Landstar's valuation ratio moves in the opposite direction of these companies' ratios, conditional on the other companies'. The ridge specification yields no coefficients with estimated value 0, but it shrinks the coefficients towards each other, so that no industry peer receives a dominant weight. Elastic net produces estimates between lasso and ridge, as one would expect given that its objective function is essentially a combination of lasso and ridge.[67]

Figure 5 plots the time series of Landstar's valuation ratio in red. The penalized regression models' predicted valuation ratios also appear in the figure. For quarter s, we compute a given model's prediction using the target-quarter valuation ratio (call this $\hat{r}_{jq}$). Thus, the penalized regression models' predicted valuation ratios are found as $\hat{r}_{jq} = \sum_i r_{is}\hat{\beta}_i$, where i indexes the industry peer firm and $\hat{\beta}_i$ is the model's estimated value of the coefficient on peer firm i's valuation ratio. The fitted ridge estimates (in purple) shrink the

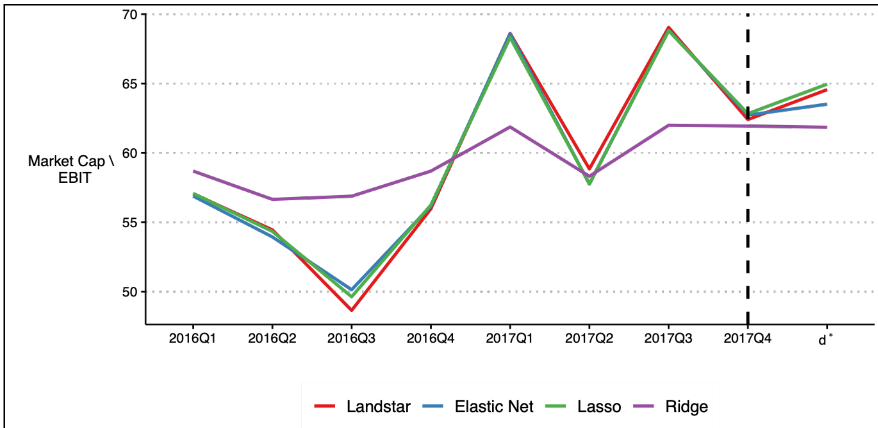
which the penalization term is proportional to the sum of squared β-coefficient values but does not depend on the sum of β-coefficient absolute-values. When $\alpha = 1$ instead, $\hat{\beta}$ is the lasso estimator, which does depend on the sum of β-coefficient absolute-values, but does not depend on the sum of squared β-coefficient values. The parameter λ governs the extent to which penalization matters; if $\lambda = 0$, then $\hat{\beta}$ is the ordinary least squares estimator.

66. We use leave-one-out cross validation to choose optimal values of the penalized-regression tuning parameter(s) in each specification.

67. In our estimates, the optimal value for the elastic net specification's parameterization for Landstar turned out to be 0.5.

fitted value toward the time-series average, smoothing away some of the time series variation in Landstar's observed valuation ratio. This smoothing is a hallmark of what penalized regression models are designed to do. The fitted lasso (in green) and elastic net (in blue) estimates are extremely close to each other, and also to the observed valuation level (again, in red). The three models produce similar target-date predictions; all are within 4.5% of the observed valuation ratio on date *d**.

Figure 5: Valuation Ratio and Penalized Regression Predictions (Landstar System Example)

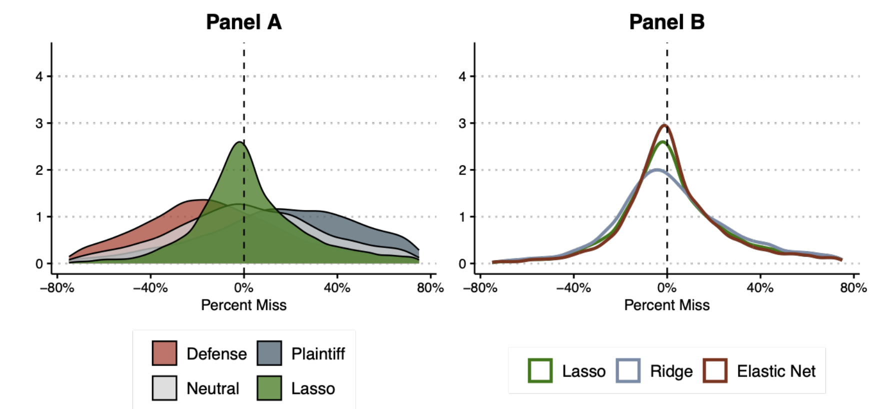


This figure reports the quarterly valuation ratio (market capitalization over EBIT) for Landstar, as well as its predicted ratio using different penalized regression models. d^* represents the randomly chosen valuation date.

The penalized regression predictions for Landstar are much closer to the observed target-date value than were most of the *k*-NN estimates plotted in Figure 1. Of course, it's impossible to determine which of these approaches is superior based on a single example, so we turn once again to results for our full set of 10,000 simulated target firm-dates.

In Panel A of Figure 6, we superimpose the density estimate of the valuation ratio predictions from the lasso specification over the plaintiff, defense, and neutral curves discussed above. Like the kernel density for the neutral *k*-NN estimates, the lasso kernel density is centered near zero. However, the lasso model performs visibly better than the neutral estimates, with noticeably lower spread. In Panel B, we plot the density estimates for all three penalized regression models. Comparing these to the *k*-NN densities in Panel A shows that all three of the penalized-regression approaches outperform the neutral *k*-NN estimates. The elastic net model's density is a bit more compressed around the 0%-deviation line than the lasso density, which in turn performs better than ridge.

Figure 6: Kernel Density Estimates Using Penalized Regression (Landstar System Example)



This figure reports the kernel density estimates for the conventional approach as well as for the penalized regression models. The reported value is measured by the percentage deviation from the observed market capitalization on the trading date d^* periods after the end of the fiscal quarter. In Panel A we report the lasso regression model estimates over the plaintiff, defense, and neutral density estimates. In Panel B, we report the three different penalized models separately.

2. *Targeting Market Cap Directly*

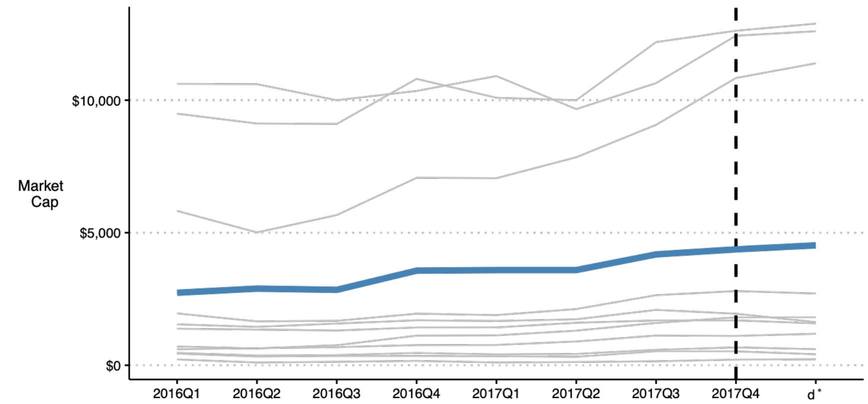
So far, all of our approaches have predicted valuation by first predicting the valuation ratio for the target firm-date or firm-quarter, and then multiplying the predicted valuation by the target firm's EBIT for the target quarter. This is a modest adaptation relative to current practice. In this subsection, we show that we can generate further performance improvements by cutting out the middle-man—the valuation ratio—and targeting market capitalization directly. It makes intuitive sense that targeting the firm valuation directly could improve prediction performance, because the goal of a valuation prediction is the market capitalization itself; the valuation ratio's role is entirely instrumental, and it seems reasonable to think that trying to predict it may introduce unnecessary noise.

To be sure, the case for targeting the valuation ratio when using traditional valuation methods is that firms that might serve as useful comps may have very different *levels* of market cap; when using a method like *k*-NN matching, some form of rescaling is necessary to guarantee reasonable estimates. However, the penalized regression approach can be used in such a way as to reduce or eliminate scale issues (e.g., by including an intercept term, as we did in the models reported in Table 4 above, and by standardizing variables before model fitting). Our next set of estimates therefore uses quarterly market cap, rather than the valuation ratio, as the dependent variable in penalized regression models.

A note of caution is warranted. All comparables-based approaches considered in this paper—like those commonly used in practice—implicitly assume that the mapping from comparable firms to the target firm remains stable after the reference period ends. If this assumption fails to hold, as may be the case in contexts such as bankruptcy valuation, the resulting estimates may be biased. In such settings, it may be appropriate to allow realized earnings to vary in the denominator of the valuation ratio after the reference period, provided one believes that the valuation mapping remains stable once changes in firm profitability are taken into account. This choice is necessarily context-dependent and turns on the facts of the case. Importantly, however, nothing in our proposal precludes incorporating post-period changes in earnings. For example, when quarterly or daily data are available, an expert may simply use the commonly reported price-to-earnings ratio, rather than market valuation, as the outcome variable.

Figure 7 plots the quarterly market capitalization value for Landstar and its industry peers for the two-year (8-quarter) period before the target date. Landstar's market cap trends upward over this period, reflecting growing firm value, as does market cap for many of its peers. However, the market capitalization values appear to be relatively less variable around their trends than were the same firms' valuation ratios.

Figure 7: Time Series Dynamics of Market Capitalization (Landstar System Example)



This figure reports the quarterly market capitalization for Landstar and its peer firms. The time series for Landstar is represented in blue while the peers are represented in gray. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

Table 5 reports the coefficient estimates from our three penalized regression models, estimated using Landstar's quarterly market cap, rather than its quarterly valuation ratio, as the dependent variable.[68] Just as before, we use the eight quarterly values ending on the randomly drawn quarter-end date (here 2017 Q4) as the right-hand side variable.

68. We note that the lasso and elastic net coefficient estimates are quite similar. This reflects the fact that the estimated optimal value of the parameter α in this sample is 0.8, very close to the lasso value of 1.

Table 5: Penalized Regression Weights on Peer Firms (Market Cap, Landstar System Example)

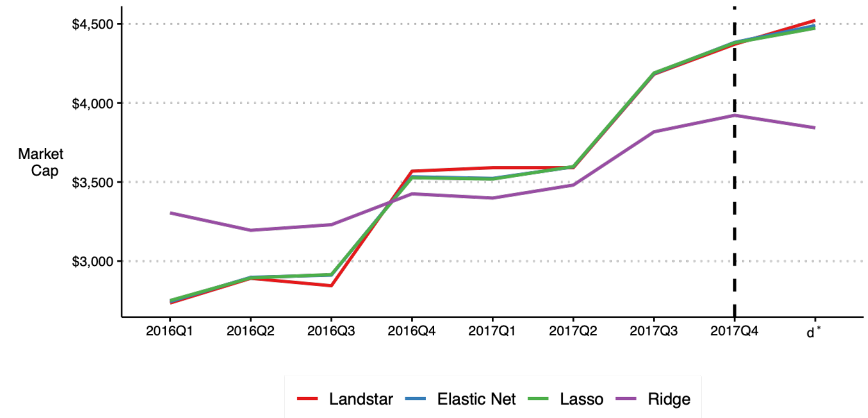
Company	Lasso	Ridge	Elastic Net
INTERCEPT	457.9128	1757.9519	383.0429
C H ROBINSON WORLDWIDE INC	0.0232	0.0211	0.0271
COVENANT LOGISTICS GROUP INC	-0.4725	0.1813	-0.5667
FORWARD AIR CORP	0.0000	0.2022	0.0000
HEARTLAND EXPRESS INC	0.0000	0.1494	0.0000
HUNT (JB) TRANSPRT SVCS INC	0.2217	0.0242	0.2304
MARTEN TRANSPORT LTD	0.0000	0.1657	0.0000
OLD DOMINION FREIGHT	0.0000	0.0171	0.0000
P.A.M. TRANSPORTATION SVCS	-2.0200	0.0114	-2.0381
SAIA INC	0.8374	0.0822	0.8225
UNIVERSAL LOGISTICS HLDGS	0.0000	0.2363	0.0000
WERNER ENTERPRISES INC	0.0000	0.0679	0.0000

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is market capitalization for the target firm, and the features that enter the regression are the market capitalizations of the peer firms. We use quarterly data for the preceding two years in fitting the model, and optimize the tuning parameter using leave-one-out cross validation.

Figure 8 replicates the analysis of Figure 7, plotting observed quarterly market capitalization for Landstar (in red) alongside the model-fitted values for each of the three penalized regression approaches.[69] The ridge regression estimates (in purple) again shrink the fitted market cap towards the sample average, while the lasso and elastic net models again produce estimates that closely match the observed valuation over the sample period. In addition, the estimates on the valuation date are much closer for the lasso and elastic net estimates.

69. For a given model, the model-based value of Landstar's market cap in quarter s is $\widehat{mc}_{js} = \sum_i mc_{is}\hat{\beta}_i$. Note that because the elastic net model produces very similar estimates to the lasso model, their lines are largely coextensive.

Figure 8: Market Capitalization and Penalized Regression Predictions (Landstar System Example)

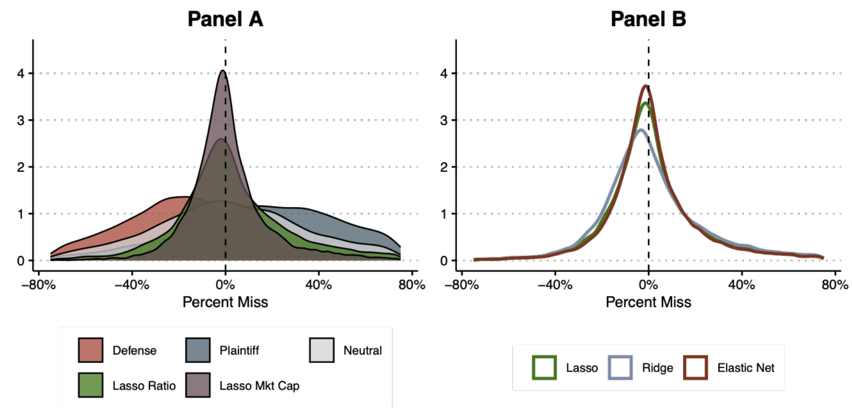


This figure reports the market capitalization for Landstar, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

Figure 9 shows how directly targeting market cap performs more generally. As with earlier kernel density figures, this one plots the density of prediction errors over our full set of 10,000 simulations of target firm-date observations. In Panel A, we add the kernel density for the lasso-market cap approach's prediction error to the densities previously plotted in Figure 6. This density is centered near the 0% prediction error line, and compared to earlier approaches, its spread is substantially reduced. Thus, the lasso-market cap approach delivers substantial improvements over the other approaches. This includes the lasso-valuation ratio approach, which Figure 6 showed had itself improved on the neutral *k*-NN approach. Panel B of Figure 9 plots the market cap-targeting kernel densities for all three penalized regression specifications, i.e., it plots the ridge and elastic net prediction error densities alongside the lasso one already included in Panel A. Although the three are relatively close, the elastic net model produces the best estimates, followed by lasso and ridge.[70]

70. Recall that elastic net involves choosing an optimal blend of the lasso and ridge estimators.

Figure 9: Kernel Density Estimates Using Penalized Regression (Landstar System Example)



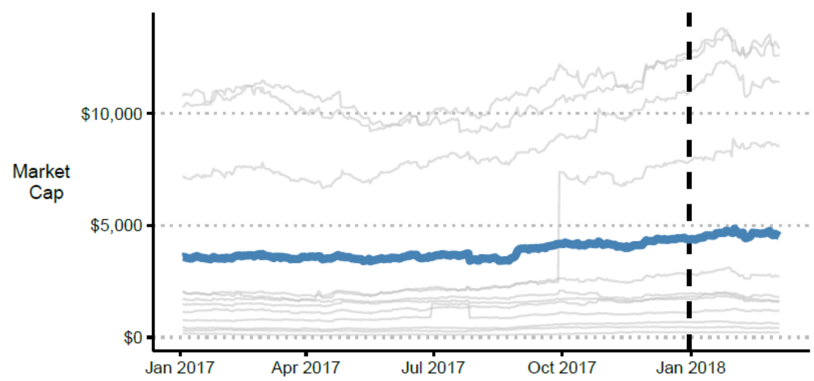
This figure reports the kernel density estimates for the conventional approach as well as for the penalized regression models. The reported value is measured by the percentage deviation from the observed market capitalization on the trading date d^* periods after the end of the fiscal quarter. In Panel A we report the lasso regression model estimates using both the ratio and market capitalization as the outcome variable over the plaintiff, defense, and neutral density estimates. In Panel B, we report the three different penalized models with market capitalization as the outcome measure separately.

3. Targeting Market Cap and Using Daily Data

So far, we have seen that there are substantial reductions in prediction error variance to be had from targeting market capitalization directly, rather than targeting the valuation ratio and then scaling by EBIT. In this section, we push further, using the fact that market cap data—unlike EBIT—can be obtained on a daily basis. The basic idea here is simple: there are many days in a quarter, so perhaps performance can be improved yet more by using daily market cap data to predict target-date firm value. Because penalized regression tends to perform better with more granular data, it seems especially natural that moving to higher-frequency data can further increase predictive performance.

Figure 10 plots the daily market capitalization values for Landstar and its industry peers for the one-year period preceding the randomly-chosen fiscal year end up to the valuation date. Figure 10 is simply the daily-measured corollary to Figure 7. We see again that Landstar's valuation trends upward over the period, and also that Landstar's market cap exceeds that of most of its peers, with comparatively stable valuation paths.

Figure 10: Daily Time Series Dynamics of Market Capitalization



This figure reports the daily market capitalization for Landstar and its peer comparables. The time series for Landstar is represented in blue while the peers are represented in gray. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter.

Table 6 reports the penalized regression model intercept and coefficient estimates, using Landstar's market cap as the dependent variable and industry peers' daily market caps as right-hand side variables. The estimation period spans the 250-day trading period ending on the target fiscal quarter end (i.e., December 31, 2017, for Landstar; note that this means that the target date, March 5, 2018, is considerably outside the estimation period).[71] These estimates are notably less sparse than those that used quarterly data, i.e., all but one industry peer gets a non-zero estimated coefficient. In addition, the estimates are similar across the three specifications (although the elastic net estimates again are closer to the lasso than to the ridge estimates).

71. Given the longer sample period in these regressions, we use 25-fold cross-validation, rather than more time-intensive leave-one-out optimization, to estimate the penalization parameters.

Table 6: Penalized Regression Weights on Peer Firms (Daily Market Cap, Landstar System Example)

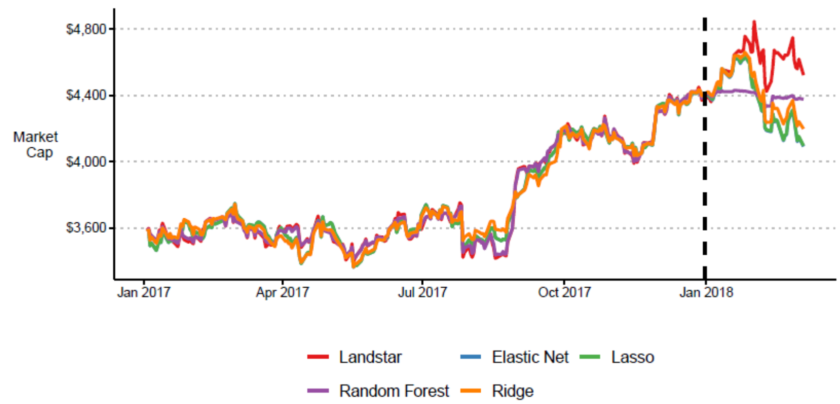
Company	Lasso	Ridge	Elastic Net
INTERCEPT	190.0837	438.0332	179.5381
C H ROBINSON WORLDWIDE INC	0.1230	0.0884	0.1254
COVENANT LOGISTICS GROUP INC	0.8335	0.5914	0.8870
FORWARD AIR CORP	0.6189	0.4711	0.6252
HEARTLAND EXPRESS INC	0.3143	0.3445	0.3167
HUNT (JB) TRANSPRT SVCS INC	-0.0207	0.0097	-0.0243
KNIGHT-SWIFT TRPTN HLDGS INC	-0.0124	0.0060	-0.0125
MARTEN TRANSPORT LTD	0.1434	0.1089	0.1494
OLD DOMINION FREIGHT	-0.0048	0.0104	-0.0078
P.A.M. TRANSPORTATION SVCS	-0.8385	-0.1693	-0.7733
SAIA INC	0.0000	0.0692	0.0000
UNIVERSAL LOGISTICS HLDGS	1.4166	0.5359	1.3550
WERNER ENTERPRISES INC	0.0392	0.0960	0.0530

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is daily market capitalization for the target firm, and the features that enter the regression are the market capitalizations of the peer firms. We use daily data starting 250 trading days before fiscal quarter end in fitting the model, and optimize the tuning parameter using 25-fold cross validation.

Figure 11 uses Table 6's coefficient estimates to predict Landstar's market capitalization over the estimation period and for the valuation date. All three penalized regression models (in green for lasso, orange for ridge, and blue for elastic net) predict Landstar's valuation (in red) almost perfectly over the estimation period, although the model-based valuations separate from the true valuation after the estimation period ends on December 31, 2017 (as one would expect). We also investigate the performance of a commonly used but less parametric approach than penalized regression, namely random forest prediction.[72] The random forest predictions also match the daily valuation of Landstar closely over the estimation period, although random forest appears to do worse out of sample (i.e., to the right of the vertical dashed line in the figure) than the penalized regression models; this is a sign of possible overfitting in the random forest predictions.

72. *See, e.g.*, Leo Breiman, *Random Forests*, 45 MACH. LEARNING 5 (2001).

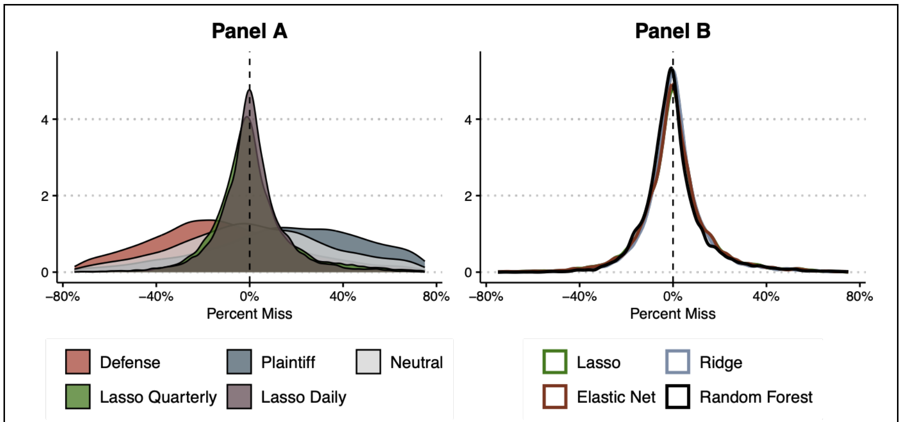
Figure 11: Daily Market Capitalization and Model Predictions (Landstar System Example)



This figure reports the daily market capitalization for Landstar, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

Turning to our full set of 10,000 target firm-dates, Figure 12's Panel A superimposes a kernel density estimate for the daily market capitalization lasso specification on the set of densities plotted in Figure 9. We see that using daily data does yield further performance improvements, although the amount of improvement seems less dramatic than earlier. Panel B separately plots the density for the random forest and the penalized regression prediction approaches. The four perform generally similarly, although the random forest and ridge regression models are best, and roughly equivalent.

Figure 12: Kernel Density Estimates Using Daily Data Penalized Regression



This figure reports the kernel density estimates for the conventional approach as well as for the penalized regression models. The reported value is measured by the percentage deviation from the true realized market capitalization on the trading date d^* periods after the end of the fiscal quarter. In Panel A, we report the lasso regression model estimates using both daily and quarterly market capitalization as the outcome variable over the plaintiff, defense, and neutral density estimates. In Panel B, we report the three different penalized models, as well as the random forest estimates, with market capitalization as the outcome measure.

4. Targeting Daily Returns for Prediction

Once we switch from quarterly to daily data, there is no obvious reason why we should continue to target the firm's market capitalization on each trading day. Because market cap is highly right skewed (i.e., there are some extreme outlier firms with very high market cap), the empirical finance literature almost exclusively uses daily stock price returns rather than stock price levels.[73] In fact, experts in the other primary area of litigation that uses stock prices—securities fraud litigation—have consistently used returns-based modeling to measure effects related to events at issue in litigation. A natural question is whether litigation involving disputed firm valuations could benefit from taking this same approach.

Let y_{it} represent the stock return for firm i on day t. We will consider five models for predicting the stock return of a target firm on a target date; we will use j to index the target firm (e.g., Landstar in our motivating example). For right-hand-side variables, we include the daily market-level return (MKTRF) as well as the daily returns from the three frequently used Fama-French-Carhart "factors": the returns on long-short portfolios sorted by size (SMB) and value (HML)[74] and the return on such portfolios sorted by momentum (UMD).[75] Baker and Gelbach (2020) show that including peer firms' returns—either via

73. *See* Stanley J. Kon, *Models of Stock Returns—A Comparison*, 39 J. FIN. 147, 147 (1984).

74. These were introduced in Eugene F. Fama & Kenneth R. French, *Size and Book-to-Market Factors in Earnings and Returns*, 50 J. FIN. 131, 143–144 (1995).

75. The UMD variable was proposed by Mark M. Carhart, *On Persistence in Mutual Fund Performance*, 52 J. FIN. 57 (1997).

an equally-weighted average or by controlling for the peers separately—can increase the predictive power of event study models, so we include their indexes as well.[76]

We consider five different models using returns as the outcome variables.

1. **FFC-Index**. We model the target firm's daily return as

$$\begin{aligned} y_{jt} \quad = \quad & \alpha + MKTRF_t\beta_1 + SMB_t\beta_2 + HML_t\beta_3 + \\ & UMD_t\beta_4 + PEERINDEX_t\beta_5 + \epsilon_{jt}, \end{aligned}$$

where $E\left[\epsilon_{jt}\middle|\cdot\right] = 0$ (here, conditioning is done on all the covariates) and $PEERINDEX_t$ is the day-t value of the return on the equally-weighted average of the target firm's industry peers.

2. **FFC-All peers**. We model the target firm's daily return as

$$\begin{aligned} y_{jt} \quad = \quad & \alpha + MKTRF_t\theta_1 + SMB_t\theta_2 + HML_t\theta_3 + \\ & UMD_t\theta_4 + \sum_i y_{it}\theta_{5i} + u_{jt}, \end{aligned}$$

where $E\left[u_{jt}\middle|\cdot\right] = 0$ (with conditioning again done on all the included covariates). Note that if $\theta_{5i} = \theta_5$, i.e., is constant over all industry peer firms i, then all θ coefficients will equal β_5 from the FFC-Index specification. We estimate the equation above using ordinary least squares (OLS).

3. **FFC-Lasso**. This approach involves lasso estimation with the target-firm's daily stock price returns, $\{y_{jt}\}_{t=1}^{T}$, on the left-hand-side and the set of right-hand-side variables including an intercept, the three Fama-French-Carhart factors, and the individual peer firm returns; penalization is done in the usual lasso way, by adding the term $\lambda \sum_h |\beta_h|$ to the usual least-squares objective function, where λ is a scalar tuning parameter and h indexes all the covariates included as explanatory variables.

4. **FFC-Ridge**. This approach involves ridge estimation with the same covariates as the lasso specification; the lasso penalization term is replaced by $\lambda \sum_h \beta_h^2$.

5. **FFC-Elastic net**. This approach is essentially a mixture of the lasso and ridge specifications, with the penalization term being $\lambda\left(\alpha \sum_h |\beta_h| + (1-\alpha)\frac{1}{2}\sum_h \beta_h^2\right)$.

All specifications use daily data for the 250 days ending on the last date of the target quarter. We estimate the FFC-Index and FFC-All Peers specifications using ordinary least squares. The last three are estimated using standard algorithms, as discussed above.[77] Table

76. *See* Andrew Baker & Jonah B. Gelbach, *Machine Learning and Predicted Returns for Event Studies in Securities Litigation*, 5 J.L. FIN. & ACCT. 231 (2020).

77. As with the daily market cap specifications, we optimize the tuning parameterizations for the penalized regression models using 25-fold cross validation.

7 reports the coefficient estimates from these five returns-based models for Landstar. The FFC-All Peers coefficients have some variation across peer firms, which indicates that the FFC-Index specification foregoes some predictive information by forcing all firms to have the same coefficient. The lasso coefficients include zeros for the value and momentum factors (HML and UMD) as well as for a number of the industry peer firms. However, the estimated elastic net value of α was 0, so that the optimal elastic net specification puts no weight on the lasso penalty; this is why the elastic net and ridge estimates are identical.

Table 7: Return Coefficients
(Daily Data, Landstar System Example)

Company	FFC Index	FFC All Peers	Lasso	Ridge	Elastic Net
INTERCEPT	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002
MKTRF	0.3768	0.1736	0.1667	0.1909	0.1909
SMB	0.1873	0.2107	0.1268	0.1527	0.1527
HML	-0.0088	-0.0224	0.0000	0.0042	0.0042
UMD	0.1063	0.0628	0.0000	0.0789	0.0789
PEER INDEX	0.5993				
C H ROBINSON WORLDWIDE INC		0.1075	0.0716	0.0833	0.0833
COVENANT LOGISTICS GROUP INC		0.0022	0.0000	0.0127	0.0127
FORWARD AIR CORP		0.0975	0.0930	0.0885	0.0885
HEARTLAND EXPRESS INC		0.0681	0.0525	0.0601	0.0601
HUNT (JB) TRANSPRT SVCS INC		0.1095	0.1010	0.0992	0.0992
KNIGHT-SWIFT TRPTN HLDGS INC		-0.0034	0.0000	0.0202	0.0202
MARTEN TRANSPORT LTD		-0.0257	0.0000	0.0224	0.0224
OLD DOMINION FREIGHT		0.2501	0.2641	0.1338	0.1338
P.A.M. TRANSPORTATION SVCS		0.0093	0.0000	0.0099	0.0099
SAIA INC		0.0443	0.0378	0.0593	0.0593
UNIVERSAL LOGISTICS HLDGS		0.0309	0.0175	0.0264	0.0264
WERNER ENTERPRISES INC		0.1133	0.1021	0.0711	0.0711

This table reports the coefficient values on the Fama-French-Carhart factors, and the peer firms, using both ordinary least squares and different forms of penalized regression. The outcome variable is the return on Landstar's stock, and the features that enter the regression are the Fama-French-Carhart factors and the returns for the peer firms. We use daily data for the 250 days prior to and ending on the fiscal-year end in fitting the model. For the penalized regression models we optimize the tuning parameter using 25-fold cross validation.

To predict valuation on the target date for each specification in Table 7, we use the same approach as above. Let h index all the explanatory variables in a specification, let Z_{iht} be firm i's value of the h^{th} covariate on date t,[78] and let $\hat{\beta}_h$ be the estimated coefficient on Z_{iht}. Then the predicted value of the target firm's daily stock price return on any date t is $\hat{y}_{jt} = \sum_i \sum_h Z_{iht}\hat{\beta}_h$. Because our objective is to predict the target firm's market cap on date d^*, we need a way to link daily returns to market cap. The target firm's market cap on date t is the product of the number of shares outstanding, S_j,[79] and the market price, P_{jt}, so the target firm's market cap on date t is $m_{jt} = S_j \times P_{jt}$. Because y_{jt} is the date-t target-firm daily return, the target firm's market price on date $t+1$ is given by $P_{jt+1} = (1 + y_{jt+1})P_{jt}$.[80] Multiplying by shares outstanding then implies that market cap on $t+1$ is given by $m_{jt+1} = S_j P_{jt+1} = S_j(1 + y_{jt+1})P_{jt}$. Using the same argument again, market cap on $t+2$ can be found by multiplying m_{jt+1} by $(1 + y_{jt+2})$. In general, the market cap

78. For the FFC factors, Z_{iht} will be the same for all peer firms on a given date.

79. We assume that the number of shares outstanding does not change between the end of the estimation period (i.e., the last date of the quarter preceding day d^*) and d^*, which is why S_j does not have a time index, t.

80. For example, if the firm has a return of $+1\%$ on $t+1$, then $y_{jt+1} = 0.01$, and the firm's price on $t+1$ equals $1.01 \times P_{jt}$.

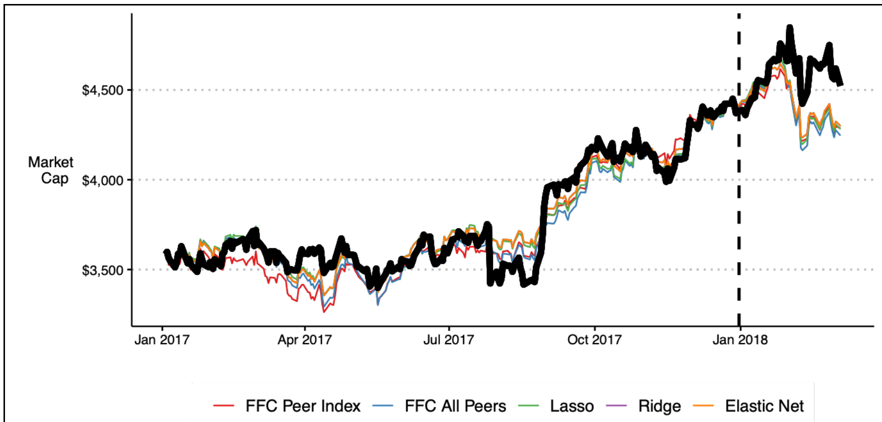
on $d^* > q$, where q is the last date of the target fiscal quarter (the quarter that precedes the target date d^*), is given by

$$\begin{aligned} m_{jd^*} &= S_j P_{jq} \prod_{d=1}^{d^*}\left(1 + y_{jq+d}\right) \\ &= m_{jq} CR_j(q, d^*), \end{aligned}$$

where $m_{jq} = S_j P_{jq}$ is the target firm's market cap on the last day of the target quarter and the term $CR_j(q, d^*) = \prod_{d=1}^{d^*}\left(1 + y_{jq+d}\right)$ is the target firm's cumulative return over the period between date q and date d^*.[81]

We report Landstar's true market capitalization (black line) and our models' predictions (various colored lines) in Figure 13. The models generally, if imperfectly, capture the trend in Landstar's market cap over this time period.

Figure 13: Daily Market Capitalization and Returns Model Predictions (Landstar System Example)



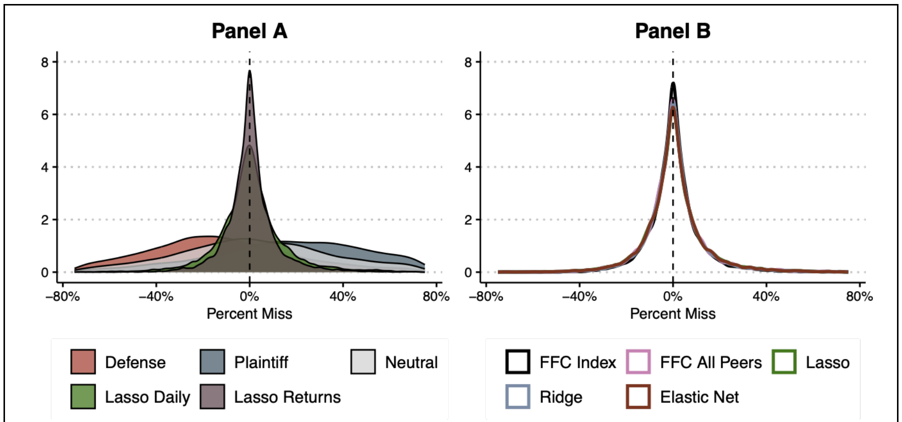
This figure reports the daily market capitalization for Landstar and the prediction from a series of models using returns rather than valuation as the outcome variable. The time series for Landstar is represented in black while the model predictions are in colors. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter.

Finally, we report kernel density estimates for the deviations of the predicted market capitalization values for our 10,000 target firm-dates in Figure 14. Panel A once again plots the prediction error densities for the previously discussed approaches, with the addition of a prediction-error kernel density for our FFC-lasso predictions. The FFC-lasso prediction density is noticeably superior to all the other densities. Interestingly, Panel B shows that the five densities involving daily returns are relatively similar-looking. Notably, *all* of them outperform even the Lasso-daily market cap density displayed in Panel A. Thus, it seems

81. In practice, this formula for the cumulative return has to be adjusted to account for any dividends paid during the (q, d^*) window. We do this appropriately in our empirical work but for expositional simplicity, we do not address the issue further in the text.

that using cumulated predicted daily returns can yield improved valuation predictions even when standard OLS estimation is used with a simple, equal-weighted industry peer index. These findings indicate that it is the use of daily returns that yields the greatest improvement in predictive accuracy relative to other methods.

Figure 14: Kernel Density Estimates Using Daily Data



This figure reports the kernel density estimates for the conventional approach as well as for the returns-based models. The reported value is measured by the percentage deviation from the observed market capitalization on the trading date d^* periods after the end of the fiscal quarter. In Panel A, we report the lasso regression model estimates using both daily market capitalization and daily returns as the outcome variable over the plaintiff, defense, and neutral density estimates. In Panel B, we report the different returns-based model estimates separately.

V. A Real-World Application: *DFC Global*

This section applies our approach to the landmark Delaware case of *DFC Global Corp. v. Muirfield Value Partners, L.P.*,[82] a shareholder dispute that metastasized into a famously focal flashpoint for valuation methodology. *DFC Global* marks something of a watershed moment in stockholder appraisal cases, which are statutorily authorized actions brought by dissenting stockholders after the close of certain eligible transactions.[83] By statutory command, the appraisal inquiry focuses on assessing the "fair value" of the target as a going concern, using "all relevant factors," and specifically excluding the value of merger synergies or takeover premiums.[84] The opposing experts in *DFC* used CC as part of their valuation analyses, and the court had to grapple with their divergent opinions.[85]

82. *See* DFC Glob. Corp. v. Muirfield Value Partners, L.P., 172 A.3d 346, 360–66 (Del. Aug. 1, 2017).

83. *See* Albert H. Choi & Eric L. Talley, *Appraising the 'Merger Price' Appraisal Rule*, 34 J.L. Econ. & Org. 543, 543–44 (2018) (offering a brief description and history of stockholder appraisal cases).

84. Del. Code Ann. tit. 8, § 262 (2024).

85. CC and DCF approaches both are part of the standard valuation canon in appraisal proceedings. *See e.g.*, HBK Master Fund L.P. v. Pivotal Software, Inc., No. 2020-0165, 2023 WL 10405169, at *22 (Del. Ch. Aug. 14, 2023) (ascribing equal weight to DCF and CC analyses).

A. Background

DFC Global is a publicly traded payday lending firm. It faced significant headwinds in 2012–13, for reasons that included its financial leverage and the regulatory scrutiny it faced in several countries. DFC engaged financial advisor Houlihan-Lockey to advise on a potential sale, and to initiate a bidding process. The bidding process was tumultuous, buffeted by several negative shocks and disappointing earnings reports, which impelled several bidders to withdraw. After contacting over 45 potential buyers, Houlihan eventually corralled 2–3 serious contenders, including a private equity company named Lone Star Funds ("Lone Star"). Ultimately, DFC signed a cash deal with Lone Star at $9.50 per share on April 1, 2014, with a closing date of June 13, 2014. Several DFC stockholders, led by hedge fund Muirfield Value Partners, pursued appraisal rights, and ultimately Chancellor Bouchard of the Delaware Court of Chancery had to determine fair value as of the closing date.[86]

B. Expert Opinions

Consistent with longstanding patterns in appraisal litigation under DGCL § 262, much of the substantive analysis in *DFC Global* came down to a valuation disagreement between opposing experts. Kevin Dages of Compass Lexecon, the petitioner's expert, performed a Discounted Cash Flow (DCF) valuation and offered an estimated fair value of $17.90 per share.[87] He also conducted a Comparable Companies (CC) analysis, choosing to use the 75th percentile among a set of ten identified comparable companies as the reference EBITDA multiple.[88] Dages reported a wide valuation interval for his CC analysis—between $11.38 and $26.95.[89] Ultimately, Dages relied entirely on his DCF estimate, giving no weight in his final opinion to the CC method, even though a detailed CC analysis was included in his report. In rationalizing this decision, Dages asserted that "[t]he reliability of a multiples-based valuation is highly dependent on the ability to identify sufficiently comparable companies and transactions, or to properly adjust financial performance data to remove non-comparable items."[90]

The respondent offered an expert report from Daniel Beaulne from Duff & Phelps.[91] Like Dages, Beaulne conducted both a DCF and a CC analysis, which yielded estimates of $7.81 per share and $8.07 per share.[92] Unlike Dages, Beaulne provided exclusively point

86. *See In re* Appraisal of DFC Glob. Corp., No. CV 10107, 2016 WL 3753123, at *12 (Del. Ch. July 8, 2016), *rev'd sub nom.* DFC Glob. Corp. v. Muirfield Value Partners, L.P., 172 A.3d 346 (Del. 2017).
87. *Id.* at *6.
88. *Id.* at *20.
89. *Id.* at *19 n.221.
90. Expert Report of Kevin F. Dages, *In re Appraisal of DFC Glob. Corp.*, Consol. C.A. No. 10107-CB (Del. Ch. 2016) at 68 [hereinafter Dages Report].
91. *In re Appraisal of DFC Global*, 2016 WL 3753123, at *6.
92. *Id.*

estimates for his valuation approaches,[93] and he gave equal weight to each of the DCF and CC estimates, so that his ultimate valuation opinion was their average, $7.94 per share.[94]

Thus, the per-share valuations offered by Beaulne and Dages were very different, with Dages valuing DFC at more than twice Beaulne's valuation. This is not an unusual situation in valuation cases.

C. The Court of Chancery Opinion

Chancellor Bouchard delivered a 68-page opinion[95] that analyzed both experts' DCF and CC analyses, as well as the deal price itself; his valuation put some weight on all three channels. As to CC, Bouchard sided with Beaulne's $8.07 figure, largely because Dages was not able to justify his 75th percentile assumption, which he had never deployed in prior valuation reports.[96] In contrast, Chancellor Bouchard substantially embraced Dages' DCF analysis, adapting it somewhat to deliver his own DCF estimate of $13.07 per share.[97] Chancellor Bouchard's opinion ultimately accorded equal weights to each of the three valuation lenses, with one-third weight apiece: DCF ($13.07/share), Comparable Companies ($8.07/share), and Deal Price ($9.50/share).[98] The end result was a blended assessment of $10.21 per share,[99] which handed a modest victory to the petitioners.[100]

D. Delaware Supreme Court's Decision

DFC appealed, making several forceful arguments. The most attention-grabbing of them was the contention that the Chancery Court's discretion should be limited in any appraisal-eligible acquisition that features an arm's-length sale following a competitive bidding process.[101] In such situations, the appellants argued, the transaction price (less synergies) should be the definitive measure of "fair value" under the appraisal statute.[102] This issue alone elicited significant attention, including dueling amicus briefs—one submitted by several law professors, and another submitted by a combined group of legal, economics,

93. An analysis of Beaulne's expert report suggests that this decision was based in part on the fact that at least one of his earnings multiples rendered a negative equity value; rather than excluding it (which would have pushed the valuation higher), he instead averaged this negative multiple with the others. *See* Expert Report of Daniel Beaulne, *In re Appraisal of DFC Glob. Corp.*, Consol. C.A. No. 10107-CB (Del. Ch. 2016) at 67–68 [hereinafter Beaulne Report].

94. *In re Appraisal of DFC Global*, 2016 WL 3753123, at *6.

95. *Id.*

96. *Id.* at *20.

97. *Id.* at *19.

98. *Id.* at *23.

99. *In re* Appraisal of DFC Glob. Corp., No. CV 10107, 2016 WL 3753123, at *23 (Del. Ch. July 8, 2016), *rev'd sub nom.* DFC Glob. Corp. v. Muirfield Value Partners, L.P., 172 A.3d 346 (Del. 2017).

100. Following Lone Star's post-hearing motion, which pointed out an error in the Court's DCF working capital projections, Chancellor Bouchard corrected his math in a revised opinion, but he simultaneously adjusted the perpetuity growth rate assumption as well, resulting in a post-correction valuation that came in at virtually the same figure as the original. The revised opinion was $10.30, slightly higher than the original. *See* DFC Glob. Corp. v. Muirfield Value Partners, L.P., 172 A.3d 346, 362 (Del. 2017) ("Taking both revisions into account, the Court of Chancery adjusted its discounted cash flow model to $13.33 per share, 2% higher than its original DCF and 40% higher than the deal price, which, when given its one-third weight, resulted in a fair value of DFC at $10.30 per share, $0.09 higher than the post-trial opinion's original award.").

101. *See DFC Glob. Corp.*, 172 A.3d at 348.

102. *Id.*

and finance.[103] In the end, the Supreme Court substantially rejected DFC's categorical argument, emphasizing the importance of preserving the Chancery Court's discretion.[104] On the issue of post-hoc adjustments in the DCF perpetuity growth rate, however, the Supreme Court held that such changes were not supported by the factual record. The Court remanded the case back to Chancellor Bouchard for reconsideration, after which the case settled.[105]

The Supreme Court's *DFC* opinion affirmed the importance of trial-court discretion and the need for a detailed explanation by the fact finder of the preferred valuation method. In some cases, a single metric may be most reliable, while in others multiple measuring perspectives should be considered. Perhaps more than anything, however, the case underscores the complexity and discretion involved in corporate valuation. That highlights the value of developing a more principled approach to determining fair value. Even if such an approach does not manage to bridge the gap in valuations between competing expert opinions, it can provide guidelines for judges considering which expert approaches are worthy of consideration.

E. A DFC Do-Over?

This section considers how DFC Global's valuation would have come out using our refined approaches to CC valuation. We treat the peer firms of DFC as the union of the potential peer firms identified in the Dages and Beaulne expert reports,[106] as well as the other financial firms in the same three-digit SIC code industry.[107]

In Figure 15, we report the daily market capitalization for DFC (blue line) and its peers (black lines) for the two-year period ending on December 31, 2013, which we use as the last "unaffected" market price preceding the appraisal action. For visualization purposes, we report the market capitalization values as percent deviations from the company's market cap as of April 1, 2012. As is evident in the data, DFC substantially underperformed most, but not all, of its industry peers over this period.

103. *See* Reynolds Holding, *DFC Global Appraisal Battle Draws Opposing Briefs from Professors*, COLUM. L. SCH.: THE CLS BLUE SKY BLOG (Feb. 7, 2017), https://clsbluesky.law.columbia.edu/2017/02/07/dfc-global-appraisal-battle-draws-opposing-briefs-from-law-professors/ [https://perma.cc/8L3X-DGJA].

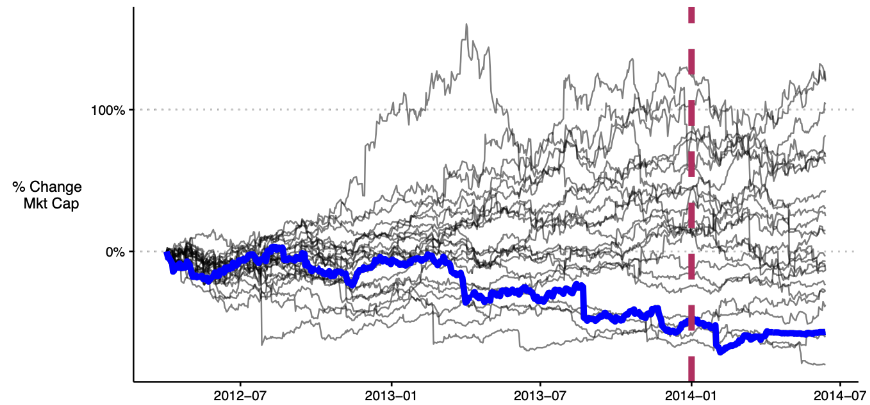
104. *DFC Glob. Corp.*, 172 A.3d at 389.

105. *Id.* at 388–89.

106. Dages Report, *supra* note 90; Beaulne Report, *supra* note 93.

107. The firm's three-digit SIC industry as of fiscal-year 2013 was 609–Functions Related to Depository Banking. We chose not to use the two-digit SIC industry definition, because 60 (Depository Institutions) covered over 500 unique firms at that time.

Figure 15: Daily Market Capitalization Over Time for DFC and Peers



This figure reports the percentage change in market capitalization for DFC and the identified peer firms for the two-year period prior to calendar year end 2013, through the valuation date. The data is measured daily.

We first ask what the predicted valuation would be if we used only *quarterly* data on market capitalization, together with penalized regression (as we did in generating the coefficient estimates reported in Part IV.B.1's Table 4). Table 8 reports the resulting lasso, ridge, and elastic net coefficient estimates for DFC Global.

Table 8: Penalized Regression Weights on Peer Firms (DFC Global Analysis)

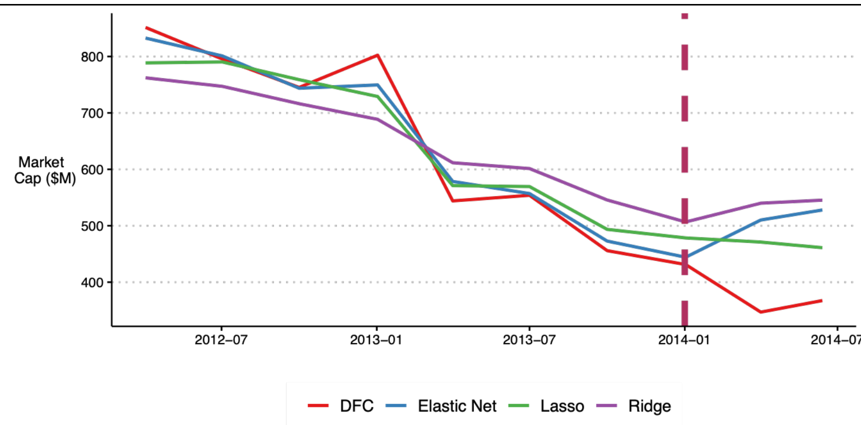
Company	Lasso	Ridge	Elastic Net
INTERCEPT	1069.7501	962.8659	1091.7664
ALBEMARLE CORP	0.0000	0.0101	0.0373
BLOCK H & R INC	-0.0406	-0.0046	-0.0147
CASH AMERICA INTL INC	0.0000	-0.0030	-0.0254
CASH CONVERTERS INTERNATIONAL LTD	0.0000	-0.0261	0.0000
CREDIT ACCEPTANCE CORP	0.0000	-0.0151	-0.0163
EURONET WORLDWIDE INC	0.0000	-0.0117	-0.0124
EZCORP INC -CL A	0.0051	0.0202	0.0174
FIRST CASH FINANCIAL SVCS	0.0000	-0.0285	-0.0532
GLOBAL CASH ACCESS HOLDINGS	0.0000	0.0037	0.0000
GLOBAL PAYMENTS INC	0.0000	-0.0093	-0.0019
GREEN DOT CORP	0.0000	-0.0171	-0.0361
HEARTLAND PAYMENT SYSTEMS	0.0000	-0.0247	-0.0415
HIGHER ONE HOLDINGS INC	0.1097	0.0408	0.1150
INTERNATIONAL PERSONAL FINANCE PLC	0.0000	-0.0131	-0.0116
MASTERCARD INC	0.0000	-0.0004	-0.0004
MONEYGRAM INTERNATIONAL INC	0.0000	-0.0343	-0.0931
PROVIDENT FINANCIAL PLC	-0.0370	-0.0159	-0.0248
QC HOLDINGS INC	0.0000	0.4198	0.1896
REGIONAL MANAGEMENT CORP	-0.4300	-0.0753	-0.1404
VERIFONE SYSTEMS INC	0.0000	0.0050	0.0109
VISA INC	0.0000	-0.0004	-0.0005
WESTERN UNION CO	0.0000	0.0012	0.0000
WORLD ACCEPTANCE CORP/DE	0.0000	-0.0624	-0.1299

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is the market capitalization for the target firm, and the features that enter the regression are the market capitalization levels for the peer firms. We use quarterly data for the preceding eight calendar quarters in fitting the model, and optimize the tuning parameter using leave-one-out cross validation. The optimal value of α in the Elastic Net model is 0.1.

Figure 16 plots the actual market valuation of DFC Global (red line) alongside the model-based predictions of market capitalization generated by the coefficient estimates in Table 8.[108] The penalized models generally do a good job at tracking DFC's valuation over this period. All three models report a valuation above the observed valuation for DFC (the red line) on the target date of June 13, 2014.

108. For reference in comparing to the Dages and Beaulne expert reports actually reported in the case, on a per-share basis, the target-date predicted valuations in the figure range from $13.59 to $15.08 per share.

Figure 16: Quarterly DFC Market Capitalization and Model Predictions (DFC Global Analysis)



This figure reports the quarterly market capitalization for DFC, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of calendar year 2013.

As we saw in Part IV.B.2, our statistical learning models performed best when we used *daily*, rather than quarterly, market data. Thus, Table 9 reports coefficient estimates from penalized regression models using data on daily rather than quarterly market capitalization, just as we did with the motivating example of Landstar in Table 6 of Part IV.B.3. The sample used to estimate the coefficients reported in Table 9 includes daily observations on market capitalization value for the one-year period ending on December 31, 2013, and as above we optimize the tuning parameters using 25-fold cross-validation.

Table 9: Penalized Regression Weights on DFC Global's Peer Firms (Based on Daily Observations)

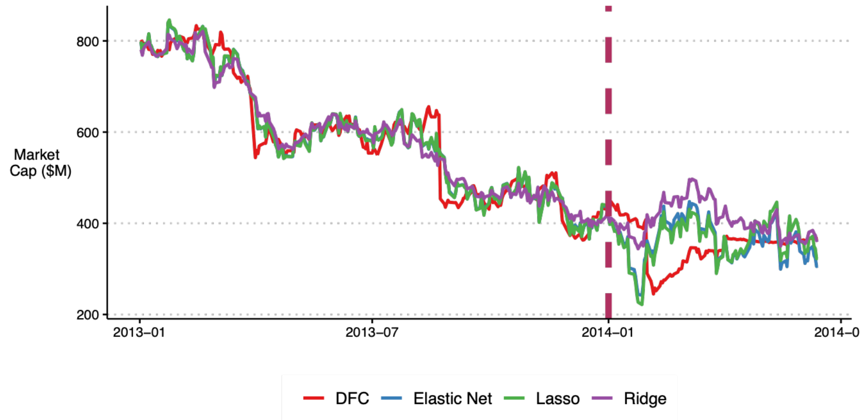
Company	Lasso	Ridge	Elastic Net
INTERCEPT	-358.0346	-43.0852	-369.1398
ALBEMARLE CORP	-0.0361	-0.0113	-0.0371
BLOCK H & R INC	-0.0019	-0.0054	-0.0027
CASH AMERICA INTL INC	0.1485	0.1027	0.1477
CASH CONVERTERS INTERNATIONAL LTD	0.0353	0.0186	0.0294
CREDIT ACCEPTANCE CORP	0.1250	0.1022	0.1211
EURONET WORLDWIDE INC	0.1496	-0.0112	0.1156
EZCORP INC -CL A	0.6163	0.2940	0.5830
FIRST CASH FINANCIAL SVCS	-0.2405	-0.1102	-0.2245
GLOBAL CASH ACCESS HOLDINGS	0.4300	0.2914	0.4675
GLOBAL PAYMENTS INC	0.0356	0.0349	0.0438
GREEN DOT CORP	-0.2711	-0.1267	-0.2613
HEARTLAND PAYMENT SYSTEMS	-0.0730	0.0047	-0.0693
HIGHER ONE HOLDINGS INC	0.7672	0.4936	0.7517
INTERNATIONAL PERSONAL FINANCE PLC	-0.0860	-0.0133	-0.0796
MASTERCARD INC	0.0052	-0.0005	0.0041
MONEYGRAM INTERNATIONAL INC	0.0179	0.0458	0.0356
PROVIDENT FINANCIAL PLC	0.0507	-0.0024	0.0486
QC HOLDINGS INC	-1.4222	1.3964	-1.1332
REGIONAL MANAGEMENT CORP	-0.3136	-0.0531	-0.1938
VERIFONE SYSTEMS INC	-0.0371	0.0028	-0.0330
VISA INC	-0.0084	-0.0037	-0.0077
WESTERN UNION CO	0.0293	0.0161	0.0281
WORLD ACCEPTANCE CORP/DE	0.1423	-0.0198	0.1261

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is the market capitalization for the target firm, and the features that enter the regression are the market capitalization levels for the peer firms. We use daily data for the one-year period ending in 2013, and optimize the tuning parameter using 25-fold cross-validation. The optimal value of α in the Elastic Net model is is 0.1

Figure 17 shows how the predictions associated with these coefficient estimates compare to the realized value for DFC over the estimation period and the valuation date. As above, the penalized regression approaches' predicted market cap largely tracks the actual market cap series over the estimation period. Moreover, the target-date predictions are quite close to DFC Global's actual market capitalization after announcement.[109]

109.The implied price per share of these predicted valuations is $6.98 to $9.74, depending on the model.

Figure 17: Daily DFC Market Capitalization and Model Predictions



This figure reports the daily market capitalization for DFC, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of calendar year 2013.

Finally, we consider using daily stock price *returns* in place of daily *market cap*, just as we did in Part IV.B.4. We report the coefficient estimates from returns-based penalized regressions in Table 10. Most potential peer firms have coefficients equal to 0 in the lasso and elastic net specifications, which are equivalent because the elastic-net's optimal value of α turns out to be 1 in this case, so that elastic net and lasso are the same. Thus, data-driven estimation rejects using most of DFC Global's industry peers.

Table 10: Coefficients for Daily Returns Approaches (DFC Global Analysis)

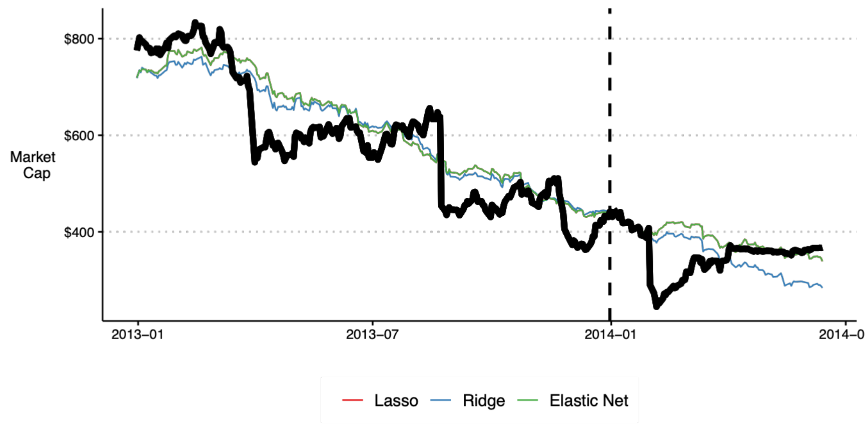
Company	Lasso	Ridge	Elastic Net
INTERCEPT	-0.0022	-0.0025	-0.0022
MKTRF	0.4596	0.2143	0.4596
SMB	0.1199	0.3112	0.1199
HML	0.0000	0.2918	0.0000
UMD	0.0000	0.0301	0.0000
ALBEMARLE CORP	0.0000	0.0629	0.0000
BLOCK H & R INC	0.0655	0.0951	0.0655
CASH AMERICA INTL INC	0.3231	0.1828	0.3231
CASH CONVERTERS INTERNATIONAL LTD	0.0000	-0.0252	0.0000
CREDIT ACCEPTANCE CORP	0.0000	-0.0032	0.0000
EURONET WORLDWIDE INC	0.0000	0.0309	0.0000
EZCORP INC -CL A	0.0000	0.0451	0.0000
FIRST CASH FINANCIAL SVCS	0.0000	0.0091	0.0000
GLOBAL CASH ACCESS HOLDINGS	0.0000	-0.0149	0.0000
GLOBAL PAYMENTS INC	0.0000	0.0195	0.0000
GREEN DOT CORP	0.0000	-0.0065	0.0000
HEARTLAND PAYMENT SYSTEMS	0.0000	0.0386	0.0000
HIGHER ONE HOLDINGS INC	0.0408	0.0619	0.0408
INTERNATIONAL PERSONAL FINANCE PLC	0.0000	-0.0223	0.0000
MASTERCARD INC	0.0000	0.0231	0.0000
MONEYGRAM INTERNATIONAL INC	0.0000	0.0301	0.0000
PROVIDENT FINANCIAL PLC	0.0000	0.0284	0.0000
QC HOLDINGS INC	0.0000	-0.0284	0.0000
REGIONAL MANAGEMENT CORP	0.0000	0.0330	0.0000
VERIFONE SYSTEMS INC	0.0000	-0.0198	0.0000
VISA INC	0.0000	0.0455	0.0000
WESTERN UNION CO	0.0000	0.0492	0.0000
WORLD ACCEPTANCE CORP/DE	0.0952	0.1094	0.0952

This table reports the coefficient values on the Fama-French-Carhart factors, and the peer firms, using different forms of penalized regression. The outcome variable is the return on DFC's stock, and the features that enter the regression are the Fama-French-Carhart factors and the returns for the peer firms. We use daily data for the year prior to and ending on December 31, 2013 in fitting the model. We optimize the tuning parameter using 25-fold cross validation. The optimal value of α in the Elastic Net model is 1.

Figure 18 plots the actual market cap for DFC Global (black line), together with the predicted values from the daily stock price returns penalized regression models. As noted, the lasso and elastic net coefficient estimates are identical, so the time series of their predicted valuations are visually indistinguishable. The results are largely consistent across penalization type, with the predicted DFC Global value on the target date equivalent to approximately $8.04 to $9.19 per share.[110]

110. We note that the pre-announcement fit is worse with the returns-based analysis, which is unsurprising given that the other methods fit directly to market value. However, as shown through our simulation analyses, these returns type analysis tend to do better on the metric that matters: post-announcement predictions.

Figure 18: DFC Daily Market Capitalization and Returns Model Predictions



This figure reports the daily market capitalization for DFC and the prediction from a series of models using returns rather than valuation as the outcome variable. The time series for DFC is represented in black while the model predictions are in colors. The vertical dashed line represents the end of calendar year 2013.

We offer Table 11 to summarize the various DFC Global valuations we discussed above. The first two blocks of rows in the table report the actual case's experts' valuations, which are roughly $8 for Beaulne (the defense's expert) and a range of roughly $11 to $27 for Dages (the plaintiff's expert). The bottom block reports our various predicted valuations using (i) quarterly market cap data, discussed in Part IV.B.2, (ii) daily market cap data, discussed in Part IV.B.3, and (iii) daily stock price return data, discussed in Part IV.B.4.[111]

Our predicted valuations range from a low of roughly $8 using daily market cap data to roughly $13 using quarterly market cap data. This is a narrower range than the range between the defense- and plaintiff-side experts in the actual case. If we were picking among these predicted valuations, we would pick the daily stock price returns approach, because that approach had the lowest prediction error in our simulation study. This means our preferred valuation would be in the neighborhood of $10—coincidentally, roughly the valuation at which Chancellor Bouchard arrived.

111. The "weighted-average" results in this table weight each of the three penalized regression model predictions by the inverse of the cross-validation squared-error. This is a sensible weighting approach because lower cross-validation error is associated with greater estimation precision.

Table 11: Fair Value Estimates – DFC

Source	Type	Estimate
Petitioner's Expert		
Dages		
	DCF	$17.90
	CC	$11.38 - $26.95
Repondent's Expert		
Beaulne		
	DCF	$7.81
	CC	$8.07
	Overall	$7.94
Data-Driven Approach		
Quarterly Market Cap		
	Lasso	$11.93
	Ridge	$14.11
	Elastic Net	$13.67
	Weighted Avg	$13.24
Daily Market Cap		
	Lasso	$8.35
	Ridge	$9.41
	Elastic Net	$7.93
	Weighted Avg	$8.61
Daily Returns		
	Lasso	$10.56
	Ridge	$9.16
	Elastic Net	$10.56
	Weighted Avg	$10.10

VI. Limitations and Extensions

The simulation results in Section 4 show that our data-driven approach likely offers substantial benefits for the practice of financial valuation in litigation. This is especially true when we are able to use daily data. Daily data might seem difficult to come by in some disputes—for example, those that center around a measure of value other than a firm's market capitalization, such as the target firm's enterprise value. The problem here is that enterprise value depends not just on market cap, but also on debt, and firms generally report aggregate debt levels only on a quarterly basis, via SEC filings (quarterly, with Form 10-Q, and annually with Form 10-K).

Ultimately, this should not be an insurmountable challenge for litigation valuation. Under bankruptcy priority rules, equity gets paid out only after debt; consequently, enterprise value calculations typically are based on the book value of debt. Thus, we can use the penalized regression approach with daily data to value equity's residual claim, and then carry forward the most recent quarterly measure of the book value of debt. Critically, this means there's no need to value firm debt on a daily basis.[112]

Another potential area of improvement in data-driven valuation is model averaging. In each case we discussed above, there are many potential valuation estimates, depending on the choice of model and data. In other settings, model-averaging has proven quite effective at capturing the inherent uncertainty in prediction exercises where the true underlying data-generating process is unknown.[113] To assess this idea, we take the predicted values using daily market cap data. There are three penalized regression predictions for each date, as well as one random forest prediction. For each of our 10,000 Monte Carlo simulations we calculate the model averaged estimate as:

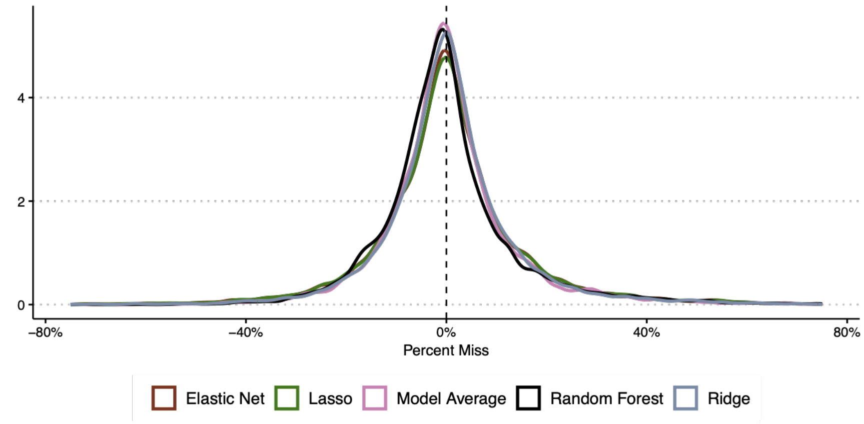
$$m = \frac{1}{4} \sum_{k} \left(w_k \times V_k \right)$$

where k denotes one of the four underlying prediction models (lasso, ridge, elastic net, and random forest), V_k is the predicted valuation for model k on date d^*, and w_k is the inverse of the estimation period cross-validation squared error. Figure 19 plots the density of the underlying model and the model average for our Monte Carlo sample. Not surprisingly given its construction, the model average does well in comparison to the underlying models. However, it does not provide visually noticeable predictive improvement over the best-performing individual models.

112. If the firm issued or redeemed an anomalous quantity of debt during the litigation-relevant period, one could value the firm's target-quarter debt level separately using the quarterly penalization approach derived above.

113. For a discussion of the use of model-averaging in the Netflix Prize, see Hal R. Varian, *Beyond Big Data*, 49 BUS. ECON. 27, 28 (2014) ("Back in 2006, NetFlix realized that 75 percent of movie views in its library were driven by recommendations. They created the 'NetFlix Prize' of $1 million that would be awarded to the group that developed the best machine learning system for recommendations, as long as it improved the current version by at least 10 percent. They provided training data of about 100M ratings, 500,000 users, and 1800 movies. A year later, the prize was won by a team that blended 800 statistical models together using model averaging.").

Figure 19: Model Averaging of Simulation Results from Part IV



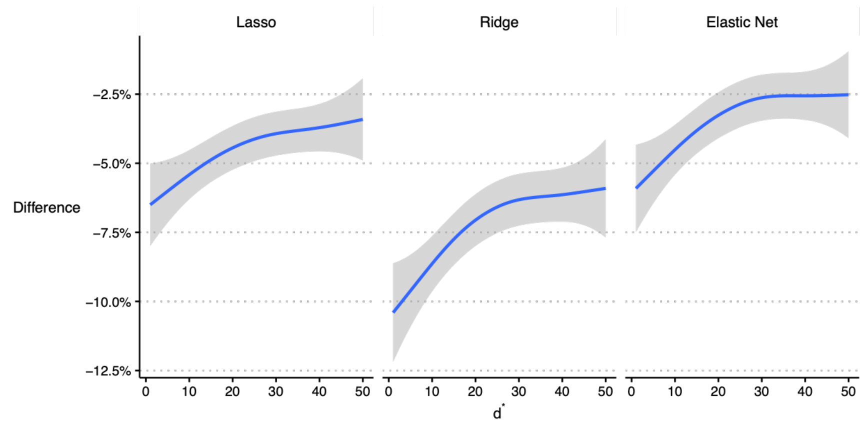
This figure reports the kernel density estimates for the machine-learning approaches to valuation using daily market capitalization as the outcome variable, as well as a model-averaged estimate. The reported value is measured by the percentage deviation from the observed market capitalization on the trading date d^* periods after the end of the fiscal quarter. The model average is the weighted average of the underlying machine learning estimates, with the inverse of the cross-validation squared error as the weight.

Finally, we note that by construction, our simulation approach partly hamstrings the performance of the models. In all cases, we restricted the end of the estimation period to q, the last date of the target quarter. This is necessary for the k-NN approach. It is unnecessary for our other approaches because daily data are available on market cap and stock price returns for the period between q and d^*. We ignored that fact here because we wanted to assess the relative performance of the various data-driven approaches when these are restricted to the same data used in k-NN prediction.

In practice, though, an expert typically will have access to daily data for dates between q and d^*, which can be included in the estimation period using daily data. That can be expected to further improve the performance of methods that use daily data. To assess this claim, Figure 20 plots the performance gains from using daily data rather than quarterly data, as a function of the length of time between quarter end, q, and valuation, d^*, for each model. To calculate performance improvement for each of the 10,000 target firm-dates, we calculate the percentage miss for the daily-data prediction (call this PM_d) and the percentage miss for the quarterly-data prediction (call this PM_q). Define $\Delta = |PM_d| - |PM_q|$; when the daily-data approach misses by more than the quarterly-data approach, regardless of the direction in which the miss occurs, Δ is positive. When instead the quarterly-data approach misses by more, Δ is negative.

Figure 20 shows that Δ is more negative for cases in which the valuation date d^* is closer to the end of the preceding quarter, q. That is, smaller gaps in dates between the end of the valuation period and the valuation date are associated with smaller percentage misses. This pattern suggests that augmenting our above approach by extending the estimation period closer to the target day could be expected to yield even more performance gains.

Figure 20: Model Improvement by d^*



This figure reports the absolute difference in the percentage miss between the daily and quarterly measured market capitalization for the penalized regression models (lasso, ridge, and elastic net) by the randomly-selected number of days post quarter-end (d^*) used as the target date. The reported curve is a smoothed generalized additive model of the difference in performance on the number of days post-quarter end, with the estimated relationship reported with a blue line, and the confidence interval for the estimate reported in grey

VII. CONCLUSION

In this paper, we critically examine the practice of financial valuation in litigation. We show that conventional methods used by experts share two serious drawbacks. First, they accord experts significant discretion to report valuations that benefit the parties the experts represent, doing so in ways that make it difficult to observe such bias directly. Second, conventional methods far underperform feasible alternatives that are now commonplace in data science. We believe the approaches we identified—most notably, predicting firm value using daily stock price returns—should be seriously considered for adoption in valuation litigation wherever possible.

Although we have limited much of our analysis to comparable companies approaches, insights developed here might also be carried over to the other two dominant approaches (comparable transactions and DCF analysis). We leave such endeavors for future work, with the hope that the contributions delivered here help provide a platform for later contributions. Additional contributions in this area could enhance the transparency, dependability, and credibility of courtroom valuation processes—something that all of us within the field would value.

DATA APPENDIX

Abbreviation	Definition	Formula	Source
age	Company Age	quarter_num - first_quarter + 1	PN
assets	Total Assets	atq	PN
capitalization_ratio	Capitalization Ratio	debt / (ev + cheq)	PN
debt	Total Debt	dlttq + dlcq	PN
debt_equity	Debt to Equity Ratio	debt + mkt_cap	PN
ebit_growth	EBIT Growth	ebit / lag(ebit, 4) - 1	PN, RP
ebit_margin	EBIT Margin	ebit / sales	RP
ebitda_growth	EBITDA Growth	ebitda / lag(ebitda, 4) - 1	PN, RP
ebitda_margin	EBITDA Margin	ebitda / sales	RP
eps_growth	Earnings Per Share Growth	oepsxq / lag(oepsxq, 4) - 1	PN, RP
ev	Enterprise Value	prccq * cshoq + pstkrq + dlcq + dlttq + mibq - cheq	PN, RP
gross_profit_margin	Gross Profit Margin	(sales - cogsq) / sales	RP
implied_dividend_yield	Implied Dividend Yield	(dvpspq * 4) / prccq	PN, RP
leverage_ratio	Leverage Ratio	debt / ebitda	PN
log_ret	Log Return	log(ret + 1)	
mkt_cap	Market Capitalization	prccq * cshoq	PN, RP
mkt_cap_ebit	Market Capitalization to EBIT Ratio	mkt_cap / ebit	PN, RP
ni_growth	Net Income Growth	niq / lag(niq, 4) - 1	PN, RP
ni_margin	Net Income Margin	niq / sales	RP
roa	Return on Assets	niq / avg_assets	PN, RP
roe	Return on Equity	niq / avg_equity	PN, RP
roic	Return on Invested Capital	ebit / avg_debt_equity	PN, RP
sales	Total Sales	saleq	RP

This table present the variables used in the conventional matching approach. It includes the formula for their construction, as well as the underlying variables as defined in Compustat and CRSP. In addition, if used as a matching variable then we report which source includes that variable in their determination set.